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# DECISION MODELLING AND OPTIMIZATION

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**LESSON 1**

**MODEL BUILDING FOR OPTIMIZATION & DISTRIBUTION AND NETWORK MODELS**

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## STRUCTURE

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## 1.1 LEARNING OBJECTIVES

After learning this chapter students would be able to formulate real life problems of logistics, networking, production, diet requirement etc. into mathematical models. It will help them to understand the practical applications of networking and theory studied would be helpful in determining the optimal solution for distribution network problems. It will be helpful in determining shortest route between any two paces, to minimize the transportation cost between two places, to maximize the efficiency of transportation between any two points etc. This lesson will make them more equipped to take better managerial decisions regarding many realistic situations discussed above.

## 1.2 INTRODUCTION

Model building for optimization is done using the technique of linear programming problem and networks. Linear Programming is a very important tool of quantitative techniques for the best possible distribution of scarce resources including labor, materials, machinery, money, energy, and so forth. It is used in almost every aspect of life, whether marketing or domestic or Production or anything else. You people are quiet familiar with the term 'Linear. It is used to describe how two or more variables in a model relate to one another proportionally. Every time a specified change in one variable occurs will always follow a given change in another variable. The term "programming" refers to device some technique for doing work in organized manner. It is a planning that involves the economic allocation of scarce resources among various options to attain the optimal goal, i.e., to maximize or minimize the objective function. Hence, Linear Programming is a quantitative technique for optimum allocation of limited or rare resources like the ones mentioned above, such as labour, materials, equip ments, money, energy, etc.
Linear programming problems in general are concerned with the use or allocation of limited resources-labour, materials, machines and capital in the best possible manner so that costs are minimized or profits are maximized. The best decision is found by solving mathematical problems. Technique of networking is used for distribution models. It includes transportation problem, perfect matching problem, maximal low problem etc. using the idea of network.
The linear programming models are widely used to solve a number of military, economic, industrial and social problems. There are various reasons for their wide uses such as:

1. A large variety of problems in diverse field can be represented as linear programming models.
2. Efficient and simple techniques are available for solving linear programming problems.

3. Data variation can be handled through linear programming models with ease.

Networking helps in determining shortest route between any two given points, It calculates the max flow, helps in determining the best assignment schedule, i.e perfect matching etc.

## 1.3 LINEAR PROGRAMMING MODEL

A Linear Programming model essentially consistsof three components.

i) The linear objective function
ii) The set of linear constraints
iii) Non-negativity of decision variables

The activities are represented by X1, X2, X3……..Xn.

These are known as Decision variables.

The objective function of a LPP (Linear Programming Problem) is a mathematical representation of the objective in terms of decision variables. It is usually a Profit or cost function. Such as

Optimize $Z = C_1X_1 + C_2X_2 + \ldots\ldots\ldots\ldots C_n X_n$ Where Z is the value of objective function, which could be maximize or minimize depending upon situation. $X_1$, $X_2$, $X_3$, $X_4$…..$X_n$ are the decision variables, and $C_1$, $C_2$, …$C_n$ are the components of cost vector that give contribution of respective decision variables.

The set of linear constraints are the set of linear equations or inequations. These constraints are mathematical expressions for the limitations under which the mathematical model is framed. For example, budget constraint, space constraint, labor constraint, time constraint etc.

### Assumptions of Linear Programming Model

All LP models assume that all constraints and objective function of model should be linear. All parameters, like the availability of resources, the unit worth of decision variable, the amount of resources, a unit of decision variable uses, are known and constant.

The optimal values of decision variables and resources are supposed to be real and non-negative numbers. If there are only whole numbers (integers) or mixed numbers (some are integers and some are fractions), the Integer programming method can be used.

The value of the objective function, given the values of the decision variables and the total amount of resources used, must be equal to the sum of the contributions (profit or cost) made by each decision variable and the sum of the resources used by each decision variable. In other words, the objective function is the direct sum of the contributions made by each variable.

General Mathematical Model of an LPP

Optimize (Maximize or Minimize)

$$Z = C_1X_1 + C_2X_2 + C_3X_3 + \ldots\ldots + C_nX_n$$

Subject to constraints,

$$a_{11}X_1 + a_{12}X_2 + \ldots\ldots\ldots + a_{1n}Xn \ (\leq, =, \geq)\ b_1$$

$$a_{21}X_1 + a_{22}X_2 + \ldots\ldots\ldots + a_{2n}Xn \ (\leq, =, \geq)\ b_2$$

$$a_{m1}X_1 + a_{m2}X_2 + \ldots\ldots + a_{mn}Xn \ (\leq, =, \geq)\ b_n$$

and $X_1, X_2 .. Xn \geq 0$

Key Points for formulating Linear Programming Model

i) Identify and define the decision variable of the problem
ii) Formulate the objective function in terms of decision variables.
iii) Formulate the constraints in terms of decision variables subject to which the objective function should be optimized (Maximization or Minimization)
iv) Add the non-negativity condition corresponding to each decision variable.

Some examples are illustrated to explain the concept.

### 1.3.1 Production Model

*Example 1*

A manufacturer produces two types of wooden toys, A and B. Each toy of the type A requires 4 hours of grinding and 2 hours of polishing, compared to 2 hours of grinding and 5 hours of polishing for each toy of type B. The producer has two grinders and three polishers. Each grinder puts in 40 hours a week of work, while each polisher puts in 60 hours. Profit on toy A is Rs. 3.00, while profit on toy B is Rs. 4.00. Everything made during a week is sold in the market. To produce the maximum profit in a week, how should the manufacturer divide up his production capacity between the two categories of toys?

Solution: Let's go step by step

i) Let $X_1$ be the number of units of toy A

$X_2$ be the number of units of toy B.

ii) As profit on each type of toy is given, therefore, the objective function is to maximize the profit, so

Max $Z = 3X_1 + 4X_2$

iii) As each toy must undergo two processes i.e. grinding and polishing. So corresponding constraints would be

$4X_1 + 2X_2 \leq 80$ (for grinding)

The number of hours for grinding machine is 40hrs per week per grinder. So total hrs for two grinders would be 80 hrs (40*2).

Similarly other constraint would be

$2X_1 + 5X_2 \leq 180$ (for polishing)

iv) By non-negativity condition

$X_1, X_2 \geq 0$

Hence Final LPP is

Max $Z = 3X_1 + 4X_2$

Subject to constraints,

$4X_1 + 2X_2 \leq 80$

$2X_1 + 5X_2 \leq 180$

$X_1, X_2 \geq 0$

### 1.3.2 Investment Model

*Example 2.*

Mr. Joshi received an amount of Rs.30,000 on his retirement, which he wishes to invest in some source from where he can get fixed income. From his friend he came to know about two types of security bonds, which yields fixed income per annum. Bond A generates 7% per annum whereas corresponding value for Bond B is 10%. Due to risk factors involved and feedback received from others, he decides to invest at most of Rs.14,000 in bond B and at least Rs.4,000 in Bond A. He also wishes that the amount investedin Bond A should not be less than the amount invested in Bond B. Formulate the LPP model for helping Mr. Joshi to generate maximum annual return from his retirement fund.

Solution

Let $X_1$ and $X_2$ be the amount invested in Bonds A and B respectively. Income generated from two Bonds are given. Hence the objective function is to maximize the income:

$$\text{Max } Z = 0.07X_1 + 0.1X_2$$

Subject to:

$$X_1 + X_2 \leq 30{,}000$$

$$X_1 \geq 4{,}000$$

$$X_2 \leq 14{,}000$$

$$X_1 \geq X_2$$

$$X_1, X_2 \geq 0$$

Now, let’s explore some minimization problems also

### 1.3.3 Cost Minimization Model

*Example 3*

A farmer uses two types of pesticides, liquid and dry for his fields. The liquid pesticide contains 6 units of chemical A, 3units of chemical B and 1 unit chemical C perjar. Respective values for dry pesticide is 1,2 and 4 units per carton. For healthy crops the minimum requirement of chemical A, B and C are 10, 12, and12 units respectively. The liquid pesticide is available in market for Rs. 40 per jar. Respective value for dry pesticide is Rs.25 per carton. How many of each pesticides should be bought in order to fulfil the requirements and keep costs down?

**Solution**

Let $X_1$ and $X_2$ be the number of units purchased of liquid anddry pesticides.

The objective function is to minimize the cost

Min. $Z = 40X_1 + 25X_2$

Subject to:

$6X_1 + X_2 \geq 10$

$3X_1 + 2X_2 \geq 12$

$X_1 + 4X_2 \geq 12$

$X_1, X_2 \geq 0$

### 1.3.4 Production Model

*Example 4*

A sewing machine manufacturer purchases semi-finished casted parts and process them to produce three different models, basic standard and premium. The casted parts undergo three different processes namely turning, milling, and drilling. The selling price of basic model is Rs 500, for standard model, it is 600 and for premium model it is Rs. 700.The demand for all the models is so large that all produced machines get sold. Cost of casted parts of three models are Rs.200, Rs.220 and Rs.250 for basic, standard, and premium models respectively.

Cost per hour to run each of the three processes are Rs.400 for Turning, Rs.375 for milling and Rs.600 for drilling. The capacities of each process for each model are shown in the following table.

| Process | Capacities Per Hour | | |
|---|---|---|---|
| | Basic | Standard | Premium |
| Turning | 25 | 40 | 25 |
| Milling | 25 | 15 | 15 |
| Drilling | 40 | 30 | 10 |

The manufacturer wants to know how many parts of each type it should produce per hour in order to maximize profit for an hour's run. Formulate this problem as an LP model to maximize total profit tothe manufacturer.

Solution:

Let $X_1$ and $X_2$ and $X_3$ be the number of basic, standard, and premium model produced per hour respectively.

With the information given, the hourly profit for basic, standard, and premium model would be as follows

Profit per unit of Basic model = (500–200) – (400/25 +375/25 + 600/40) = 254

Profit per unit of standard model = (600-220) – (400/40 + 375/15 + 600/30) = 325

Profit per unit of premium model = (700 – 250) – (400/25 + 375/15 + 600/10) = 349

Hence Objective Function is

Maximize $Z = 254 X_1 + 325X_2 + 349X_3$

Subjected to:

$X_1/25 + X_2/40 + X_3/25 \leq 1$

$X_1/25 + X_2/15 + X_3/15 \leq 1$

$X_1/40 + X_2/30 + X_3/10 \leq 1$

$X_1$, X2, $X_3 \geq 0$

1.3.5 Man Power Scheduling

*Example 5*

Apollo hospital needs different number of nursing staff at different timings of day. Each nurse has a duty of 8 hrs in a day and reports at the beginning of period. The hospital management wants to formulate the plan that how many nurses should be called to meet the daily needs. The following table summarizes the number of nurses needed round the clock.

| Period | Clock time (24 hours day) | Minimum number of nurses required |
|---|---|---|
| 1 | 8 a.m. – 12 noon | 3 |
| 2 | 12 noon. – 4 p.m. | 6 |
| 3 | 4 p.m. – 8 p.m. | 14 |
| 4 | 8 p.m. – 12 mid night | 6 |
| 5 | 12 mid night – 4 a.m. | 10 |
| 6 | 4 a.m. – 8 a.m. | 8 |

In order to have enough nurses available during each period, the hospital seeks to determine the bare minimum number of nurses that should be employed. Formulate the situation as a linear programming problem.

Solution

Let $X_1$, $X_2$, $X_3$, $X_4$, $X_5$ and $X_6$ be the number of nurses joining duty at the beginning of periods 1, 2, 3, 4, 5 and 6respectively.

Objective function is

Minimize $Z = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$

Subject to

$X_1 + X_2 \geq 6$

$X_2 + X_3 \geq 14$

$X_3 + X_4 \geq 6$

$X_4 + X_5 \geq 10$

$X_5 + X_6 \geq 8$

$X_6 + X_1 \geq 3$

$X_1, X_2, X_3, X_4, X_5, X_6 \geq 0$

1.3.6 Trim loss Problem

Example 6: The Modern Paper Mart produces paper rolls with a standard width of 20 m each. The company receives orders of different width. These widths are produced by slitting the rolls of 20 m. The typical order received on one day is summarized as follows:

| Order | Desired Width (m) | Desired number of Rolls |
|---|---|---|
| 1 | 5 | 150 |
| 2 | 7 | 200 |
| 3 | 9 | 300 |

Formulate the above problem as a linear programming problem to meet the order with minimum trim loss.

Solution: 20m width roll can be cut according to following combinations to meet the required order:

| S.No. | 5m | 7m | 9m | Trim Loss (m) |
|---|---|---|---|---|
| **1** | 4 | 0 | 0 | 0 |
| **2** | 2 | 1 | 0 | 3 |
| **3** | 2 | 0 | 1 | 1 |
| **4** | 1 | 2 | 0 | 1 |
| **5** | 0 | 1 | 1 | 4 |
| **6** | 0 | 0 | 2 | 2 |

Let $X_i$ , i=1,2…,6 be the number of rolls cut according to $i^{th}$ combination.

Objective function is

Minimize $Z = 0X_1 + 3X_2 + 1X_3 + 1X_4 + 4X_5 + 2X_6$

Subject to

$4X_1 + 2X_2 + 2X_3 + 1X_4 >= 150$

$X_2 + 2X_4 + X_5 >= 200$

$X_3 + X_5 + 2X_6 >= 300$

$X_1, X_2, X_3, X_4, X_5, X_6 >= 0$

**In- Text Questions**

Section 1.3

Q1. What are the essential components of linear programming model.

Q2. How minimization problem can be converted into maximization problem?

Q3. Why linear programming problems are so popular?

Q4. Can objective function and/or constraints of LPP be non-linear?

## 1.4 DISTRIBUTION AND NETWORKING MODELS

Linear programming can be applied distribution and networking models also. Transportation is one of such problems. Under this category we consider the physical transport of goods from multiple sources to multiple destinations. Each source has some limited availability and each destination has some particular demand. The objective of the problem is to find the schedule that how many items should be transported from a particular source to particular destination so as to satisfy the demand of each destination with the available units at minimum cost of transportation.

Above situation can be explained diagrammatically as Sources Destinations

This problem can be converted into Linear Programming Problem as follows:

Let$X_{ij} >= 0$ be the number of units to be sent from $S_i$ to $D_j$ per unit time. Now the objective is minimizing the transportation cost under the availability and demand constraint, so objective function is

$$\text{Min } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}$$

Where Cij is the cost of transportation from ith source to jth destination subject to the constraints that the Sum of the quantity of goods transported to different destinations from source i must be less than equal to the availability of source i that is

$$\sum_{j=1}^{n} x_{ij} \le s_i \quad \text{for } i = 1, \ldots, m,$$

and the the Sum of the quantity of goods transported to jth destination must be greater than demand $d_j$ , that is

$$\sum_{i=1}^{m} x_{ij} \ge d_j \quad \text{for } j = 1, \ldots, n.$$

The necessary and sufficient condition for the existence of solution is that

Total supply = total demand

We thus define the transportation (or Hitchcock) problem as the following LP, where the $s_i >= 0$, $d_j >= 0$, $c_{ij} >=0$ are given, with Total supply = total demand

Hence, Linear programming problem is

$$\text{Min} \sum_{i,j} c_{ij} x_{ij}$$

subject to

$$\sum_{j} x_{ij} = s_i \quad \text{for each } i = 1, \ldots, m$$

$$\sum_{i} x_{ij} = d_j \quad \text{for each } j = 1, \ldots, n$$

$$x_{ij} \ge 0 \text{ for each } i = 1, \ldots, m \text{ and } j = 1, \ldots, n.$$

Which can be solved using the method of IntegerProgramming.

But as it is a separate class of problem we have special method also for solving this type of problem.

1.4.1 Transportation Problem

Solution procedure for solving Transportation Problem.

- Define the objective function to be minimized
- Set up the transportation table with n rows representing the destinations and m columns representing the sources
- At (i,j) position of this table write the unit transportation cost from jth source to ith destination.
- Write the availability each source and demand each destination(as given in following lay out)
- Now check whether the total supply = Total demand or not. If yes then problem is balanced otherwise balance it by adding a dummy row or column of difference to make it balanced.
- Find initial basic feasible solution
- Test whether it is optimal or not . If optimal then stop otherwise improve the solution and again check for optimality until you reach optimal solution.

| Source\Destination | D1 | D2 | D3 | Availability |
|---|---|---|---|---|
| S1 | | C12 | | A1 |
| S2 | | | | A2 |
| S3 | | | C31 | A3 |
| S4 | | | | A4 |
| Demand | d1 | d2 | d3 | |

In above table C12 shows cost of transporting one unit from source 1 to destination2 and similarly C31 shows cost of transporting one unit from source 3 to destination1.

**Conversion Of Unbalanced problem to Balanced Problem**

If Total availability is not equal to total demand, then such a problem is known as unbalanced problem. The very first condition for the problem to be solvable is that It must be balanced. So, let's take an example to see how an unbalanced problem can be converted into balanced one.

| Source\Destination | D1 | D2 | D3 | Availability |
|---|---|---|---|---|
| S1 | | C12 | | 8 |
| S2 | | | | 10 |
| S3 | | | C31 | 10 |
| S4 | | | | 12 |
| Demand | 10 | 10 | 5 | |

Total Demand = 25Total Supply =40

As the total supply is greater than total demand therefore given problem is unbalanced. To make it balanced add a dummy destination with demand equal to (40-25) ie 15. For this destination each cost would remain zero as this is not a real demand.

**Methods For Obtaining Basic Feasible Solution (BFS)**

An initial BFS can be obtained by any of followingThree methods:

1. North- West Corner Rule
2. Least Cost Matrix Method
3. Vogel's Approximation Method (VAM)

**1.** North–west corner rule

- Write the given transportation problem in tabularform
- Check whether the problem is balanced or not. Ifnot then make it balanced and go to next step.
- Go to the north-west corner of the table. Supposeit is the (i, j)$^{th}$ cell.
- Allocate min ($Ai_i$, dj) to this cell. If the min ($Ai_i$, dj) = $A_i$, then the availability of the $i^{th}$ origin is exhausted and demand at the $j^{th}$ destination remains as $d_j$-$a_i$ and the $i^{th}$ row is deleted from the table. But if min ($Ai_i$, dj) = $d_j$, then demand at the $j^{th}$ destination is fulfilled and the availability at the $i^{th}$ origin remains to be $A_i$-$d_j$ and the $j^{th}$ column is deleted from the table.
- Repeat above 2 steps until all availabilities get exhausted and demands are fulfilled

| To<br>From | D1 | D2 | D3 | D4 | Capacity |
|---|---|---|---|---|---|
| **S1** | 21 | 16 | 25 | 13<br>11 | 11 |
| **S2** | 17 | 18 | 14<br>9 | 23<br>4 | 13 9 |
| **S3** | 32<br>6 | 27<br>10 | 18<br>3 | 41 | 19 16 6 |
| **Demand** | 6 | 10 | 12 3 | 15 4 | 43 |

Hence, Initial BFS is

$X_{14} = 11$ $X_{23} = 9$ $X_{24} = 4$ $X_{31} = 6$ $X_{32} = 10$ $X_{33} = 3$

## 2. Least Cost Matrix Method (LCMM)

In this method, allocations are made on the basis of economic desirability. The steps involved in determining an initial solution using least-cost method are as follows:

- Write the given transportation problem in tabular form.
- Choose the cell with minimum cost. If it is not unique, anyone can be chosen. Suppose it is the (i, j) th cell.
- Allocate min ($A_{ij}$, $d_j$) to this cell. If the min ($A_{ij}$, $d_j$) $= A_i$, then the availability of the origin is exhausted and demand at the $j^{th}$ destination remains as $d_j - A_i$ and the ith row is deleted from the table. But if min ($A_{ij}$, $d_j$) = dj, then demand at the jth destination is fulfilled and the availability at the ith origin remains to be Ai-dj and the jth column is deleted from the table.
- Repeat above 2 steps until all availabilities get exhausted and demands are fulfilled.

| To<br><br>From | D1 | D2 | D3 | D4 | Capacity |
|---|---|---|---|---|---|
| S1 | 21 | 16 | 25 | 13<br>11 | 11 |
| S2 | 17<br>1 | 18 | 14<br>12 | 23 | 13 1 |
| S3 | 32<br>5 | 27<br>10 | 18 | 41<br>4 | 19 9 4 |
| Demand | 6 5 | 10 | 12 | 15 4 | 43 |



## Vogel's Approximation Method (VAM)

The idea of limiting opportunity (or penalty) costs forms the foundation of VAM. The difference between the two least expensive choices for any source and destination is known as the opportunity cost for that source or destination. This penalty suggests that any unsatisfied demand or supply would be satisfied at an additional cost per unit, which would be equal to an opportunity cost or penalty. This approach is chosen over the others mentioned above since it requires less iterations to reach the ideal result. The most effective strategy for locating an initial, workable basic answer is VAM. The following procedures are used to determine an initial solution while employing VAM.

- • Create a matrix table for the provided transportation problem.
- Calculate the penalty cost, which is the difference between the minimum cost and the next lowest cost for each row and each column.
- Regardless of row or column, select the highest difference or penalty cost. Choose the row/column cell with the lowest cost and begin allocating to it. Suppose it is the (i, j)th cell.
- Next to this cell allocate min ($A_{ij}$, $d_j$). If the min ($A_{ij}$, $d_j$) = Ai, then the availability of the ith origin is exhausted and demand at the jth destination remains as dj-ai and the ith row is deleted from the table. But if min ($A_{ij}$, $d_j$) = dj, then demand at the jth destination is fulfilled and the availability at the ith origin remains to be Ai-dj and the jth column is deleted from the table.

- Repeat above 2 steps until all availabilities get exhausted and demands are fulfilled.

Note:

- It is always advisable to use VAM as initial basic feasible solution, if method is not specified. The reason behind it is very simple that this provides solution which is very close to optimal solution.
- If at any point before the end, a row's supply and column's demand are both satisfied simultaneously, then both will be crossed out and the next variable to be added to the basic solution will necessary be at the zero level. Such a situation is known as degeneracy.

| To<br><br>From | D1 | D2 | D3 | D4 | Capacit<br>y | Ui |
|---|---|---|---|---|---|---|
| S1 | 21 | 16 | 25 | 13<br>11 | 11 | 3 |
| S2 | 17<br>6 | 18<br>3 | 14 | 23<br>4 | 13 9 3 | 3 3 4 |
| S3 | 32 | 27<br>7 | 18<br>12 | 41 | 19 7 | 9 9 9 |
| Demand | 6 | 10 | 12 | 15 4 | 43 | |
| Vj | 4<br>15 | 2<br>9<br>9 | 4<br>4<br>4 | 10<br>18 | | |

## Optimal Solution

Once an initial basic feasible solution has been found, we will move towards the technique to find optimal solution. Following two techniques are used to improve BFS

- Stepping Stone Method
- Modified Distribution Method (MODI). Here we are discussing only MODI method.

**Stepping Stone Method**

- Determine the initial basic feasible solution by any of the three methods defined above.
- Draw closed loop, starting from any non-basic cell. Loop should be drawn with horizontal or vertical lines in such a way that corner should come only at basic cells and loop should end at same non-basic cell from where it was started. Mark alternatively (+) and (-) sign at the corners of the loop traced, starting from non-basic cell respectively. Calculate the transportation cost by adding or subtracting the cost of each corner depending upon marked sign. This cost is identified as net cost change between initial basic feasible solution and new solution. Repeat the procedure to calculate net cost change corresponding to each non-basic cell.
- If net cost change corresponding to each non-basic cell is positive, then initial basic feasible solution is optimal. Otherwise, select the non-basic cell corresponding to which highest negative net cost change is obtained.
- Out of all (-) marked cells, select minimum allocation. Add this value at (+) marked corners and subtract at (-) marked corners. This would be new basic feasible solution. Again, calculate net cost change corresponding to each non-basic cell and check for optimality.
- Repeat the above steps till the condition of optimality is satisfied.

**Modified Distribution Method (MODI)**

- Determine the initial basic feasible solution by any of the three methods defined above.
- If number of Basic cells (All those cells which have allocation) are m+n-1, then go to next step, otherwise give zero allocations to some non-basic cells so as number of basic cell becomes m+n-1 .

- Determine the set of numbers Ui (for rows) and Vj (for column) in such a way that for Basic Cells Ui+Vj-Cij=0.
- Calculate the value of Ui+Vj-Cij (opportunity cost) for each non-basic cell. If opportunity cost for each non-basic cell is negative or zero then the current solution is optimal. otherwise the non-basic cell which has highest opportunity cost must be entered into the basis and one of basic cell wll become non- basic.
- Repeat the above two steps till the condition of optimality is satisfied.

### 1.4.2 Assignment problem

It is a special case of transportation model where workers and jobs represent the sources and destinations respectively. The supply (demand) at each source (destination) is exactly equal to 1. The cost of transportation Cij represents the wages given to worker i if jth job is assigned to him/her. Such problem can be formulated as linear programming problem just as transportation problem as explained in previous section. These problems can be solved using Hungarian method. The basic assumptions of Hungarian Method are explained in next section.

**Basic Assumptions:**

- Only one job can be assigned to each worker and only one worker can be assigned to each job.
- Number of jobs should be equal to number of workers. i.e Problem should be balanced.
- If problem is unbalanced, then by adding dummy jobs or dummy workers with 0 as associated cost problem must be balanced first.
- Problem should be a minimization problem.
- If problem is a maximization problem, then convert it into minimization first.

**Hungarian Method**

- Check whether the problem is balanced or not, if not convert into balanced problem.
- Check whether it is a minimization problem or not, if not convert it into a minimization problem.
- Select the row minima from each row and subtract it from its corresponding row, the technique is termed as row reduction method.

- Select the column minima from each column and subtract it from its corresponding column, the technique is termed as column reduction method.
- Start assignment by selecting Zeroes from rows/ column in such a way that there should be single assignment in each row or column.
- If each row and column get assignment, then the current solution is optimal. Calculate the cost of assignment by adding the corresponding values in original matrix.
- If any row/column is left where there is no assignment then follow the improvement schedule to improve the table and again do assignment.

**Conversion of Unbalanced Problem to Balanced Problem**

| Workers ® Job – | W1 | W2 | W3 | W4 |
|---|---|---|---|---|
| J1 | 6 | 4 | 8 | 6 |
| J2 | 8 | 5 | 2 | 4 |
| J3 | 9 | 4 | 7 | 3 |

In above problem we have 3 jobs but 4 workers, so to make it balanced add a dummy job with all associated cost as zero.

| Workers ® Job – | W1 | W2 | W3 | W4 |
|---|---|---|---|---|
| J1 | 6 | 4 | 8 | 6 |
| J2 | 8 | 5 | 2 | 4 |
| J3 | 9 | 4 | 7 | 3 |
| J4 (Dummy) | 0 | 0 | 0 | 0 |

**Conversion of Maximization to Minimization**

Let the above matrix is representing the profits earned by the firm when ith job is done by the jth worker. Now, Firm wants to allocate the jobs to the workers in such a way, so that profit can be maximized. It can be converted into minimization problem by following algorithm.

- Select the maximum of matrix i.e "9".
- Subtract the entries of whole matrix from this 9
- Use the reduced matrix for solving the problem.

Reduced matrix is:

| Workers ® Job | W1 | W2 | W3 | W4 |
|---|---|---|---|---|
| J1 | 3 | 5 | 1 | 3 |
| J2 | 1 | 4 | 7 | 5 |
| J3 | 0 | 5 | 2 | 6 |
| J4 (Dummy) | 9 | 9 | 9 | 9 |

After assignment the values of assigned positions would be added from the original matrix to get the maximum profit.

*Example:*

ABC Transco has four trucks namely A, B, C and D and four sites The numbers given in the following table shows the distance in km associated with each pair of truck and site. Find the assignment schedule for the following problem in order to minimize the total distance (in km) travelled?

| Trucks Sites | A | B | C | D |
|---|---|---|---|---|
| 1 | 90 | 75 | 75 | 80 |
| 2 | 35 | 85 | 55 | 65 |
| 3 | 125 | 95 | 90 | 105 |
| 4 | 45 | 110 | 95 | 115 |

**Solution:** Subtract 75 from Row 1, 35 from Row 2, 90 from Row 3, and 45 from Row 4.

| Trucks Sites | A | B | C | D |
|---|---|---|---|---|
| 1 | 15 | 0 | 0 | 5 |
| 2 | 0 | 50 | 20 | 30 |
| 3 | 35 | 5 | 0 | 15 |
| 4 | 0 | 65 | 50 | 70 |

**Subtract 0 from Column 1, 0 from Colum 2, 0 from Column 3, and 5 from Column 4**

| Trucks ® Sites _ | A | B | C | D |
|---|---|---|---|---|
| 1 | 15 | 0 | 0 | 0 |
| 2 | 0 | 50 | 20 | 25 |
| 3 | 35 | 5 | 0 | 10 |
| 4 | 0 | 65 | 50 | 65 |

Start assignment

| Trucks ® Sites _ | A | B | C | D |
|---|---|---|---|---|
| 1 | 15 | | 0 | 0 |
| 2 | 0 | 50 | 20 | 25 |
| 3 | 35 | 5 | | 10 |
| 4 | | 65 | 50 | 65 |

**After assignment we get just three assignments soimprove solution.**

- Draw minimum number of lines to cover all thezeroes.
- Select the minimum of uncovered element*, and subtract it from uncovered

and add it to doublycovered** and do nothing for singly covered***.

- Again start assignment, if all four assignments obtained, then optimal otherwise repeat the steps from beginning.

| Trucks ® Sites – | A | B | C | D |
|---|---|---|---|---|
| 1 | 15 | | 0 | 0 |
| 2 | 0 | 50 | 20 | 25 |
| 3 | 35 | 5 | | 10 |
| 4 | | 65 | 50 | 65 |

Minimum uncovered element is 5, Hence new matrix is

| Trucks | A | B | C | D |
|---|---|---|---|---|
| 1 | 20 | | 5 | 0 |
| 2 | | 45 | 20 | 20 |
| 3 | 35 | 0 | | 5 |
| 4 | 0 | 60 | 50 | 60 |

* those elements which are not crossed by lines

** those elements which lie at intersection of two lines

*** those elements which are under a single line.

**Minimum uncovered is 20, so new matrix is**

| Trucks ® Sites | A | B | C | D |
|---|---|---|---|---|
| 1 | 40 | | 5 | 0 |
| 2 | 0 | 25 | 0 | |
| 3 | 55 | 0 | | 5 |
| 4 | | 40 | 30 | 40 |

Hence the optimal assignment is

Truck 1 to Site B Truck 2 to Site D Truck 3 to Site C and Truck 4 to Site A.

At cost of 75+65+90+45=275 km



### 1.4.3 Shortest route Problem

Shortest route problem is defined as a problem of finding the shortest path between the desired nodes of a given graph. The graph could be graphical representation of road network, communication network, logistic network etc. The most popular method of finding shortest path between any node (source) to every other node is Dijkstra's algorithm. Before explaining Dijkstra's algorithm lets understand graph and some basic terms of graph.

The diagrammatical representation of explaining the connectivity between different elements of some network is called graph. These elements are called nodes and the arcs by which these elements are connected are called edges. The node, from which distance is calculated is called source and the node till which distance is calculated is called sink. For example, in case of road maps, the different cities are called nodes and the roads by which these cities are connected are called the edges.

**Dijkstra's Algorithm**

- It begins at source and examines the graph to determine the shortest route between source and every other node of network.

- The algorithm maintains a record of the presently known shortest distance between source and every other node, and it changes these values whenever it discovers a route that is shorter than the previously known shortest distance.
- After the algorithm has determined the route that is the shortest between the source node and another node, the algorithm updates label on the other node as either "visited" or “permanent” from “unvisited” or “temporary” and adds it to the path. This path initially contains only source node and one by one other nodes get added to it.
- The procedure is repeated until each node in the network has been included in the route. It means procedure terminates when label of each node becomes "visited" or “permanent” . At the end we get a route that connects the source node to all of the other nodes with shortest path length between them.

**Example:** Consider the following graph with six nodes. The numbers written on edges expresses the distance between corresponding nodes. Use Dijkstra’s algorithm to find the shortest distance of each node from node S.

**Solution:** Initially Only S is labelled as “visited” and rest all five nodes are labelled as “unvisited”. Also write the label on each node along with distance and the name of node from which that distance is measured. To maintain the record lets write in tabular form.

| Iteration | Visited or permanent node | Unvisited or temporary node | Label(distance, preceding node) |
|---|---|---|---|
| 1 | S | A | (1,S) |
| | | B | (5,S) |
| | | C | (∞,S) Since no path between 3 and S |
| | | D | (∞,S) Since no path between 4 and S |
| | | E | (∞,S) Since no path between 5 and S |

Now, Nodes A and B can be traced from node S in 1 unit of distance and 5 units of distance respectively. Since min(1,5)=1

Hence, node A would be traced from node S and node A is labelled as “visited”. Now path becomes {S,A} at shortest distance of 1 unit.

| Iteration | Visited or permanent node | Unvisited or temporary node | Label(distance, preceding node) |
|---|---|---|---|
| 2 | S, A | B | (1+2,A) |
| | | C | (1+2,A) |
| | | D | (1+1,A) |
| | | E | (∞,S) Since no path between 5 and S through visited nodes |

Label for node B Could either be (5,S) or (1+2,A)=(3,A). Since distance of node B via node A is less hence its label would be (3,A) not (5,S). Which means shortest distance of node B from S is 3 units via node A. Similarly labels for rest of the nodes are written. At this point node D has least distance among unvisited nodes, so label of node D updated to visited.

| Iteration | Visited or permanent node | Unvisited or temporary node | Label(distance, preceding node) |
|---|---|---|---|
| 3 | S,A,D | B | (1+2,A) |
| | | C | (1+2,A) |
| | | E | (∞,S) |

Since node B and C can be visited from node A at the distance of unit 3, which is minimum too amongst all unvisited nodes, therefore both become visited now.

| Iteration | Visited or permanent node | Unvisited or temporary node | Label(distance, preceding node) |
|---|---|---|---|
| 4 | S,A,D,B,C | E | (3+1,C) or (2+2,D) |

Node E could be traversed either from node C or from node D as both length are same, ie 4 units. Hence shortest path of each node from node S is

S to A, 1 unit with path SA

S to B, 3 units with path SAB

S to C, 3 units with path SAC

S to D, 2 units with path SAD

S to E, 4 units with path SACE or SADE

### 1.4.4 Maximal Flow problem

The objective of the max flow problem is to find the greatest amount of flow that can be sent through a network of pipelines, channels, or other passageways while capacity limitations is taking into account. This is one of the main concerns handled via graph theory. The problem can be used to simulate a broad variety of real-world circumstances, including resource distribution, communication networks, and transportation systems, to name a few.

In the maximum flow problem, we have a directed graph with a source node S and a sink node T, and each edge has a capacity that symbolizes the maximum amount of flow that can be sent through it. In other words, the maximum amount of flow that can be sent through an edge is called its capacity. The objective here is to determine the greatest quantity of flow that can be transmitted from point S to point T while still adhering to the capacity limitations imposed by the edges. The most common algorithm for solving maximal flow problem is Ford-Fulkerson algorithm.

**Ford-Fulkerson algorithm**

This algorithm is based on finding the flow- augmenting path, residual capacity of each edge from source to sink and bottleneck capacity of augmenting path.

Residual Capacity - Residual capacity of the directed edge is defined as the remaining capacity of the edge. i.e, original capacity of the edge - current flow through the edge. If there is flow along a directed edge $u \rightarrow v$ then reversed edge has a capacity 0 and it can be considered like

$$f(v,u)=-f(u,v)$$

Residual Graph – The original graph in which instead of original capacities, Residual capacities as written on edges.

Augmenting Path – The series of edges or the path from source to sink in residual graph is called augmenting path.

Bottleneck capacity- the minimum of residual capacities on any augmenting path is called bottleneck capacity.

Let's explain the algorithm with the help of an example.

**Example:** Consider the following network, where the numbers given on arcs are capacities of corresponding edges. Find the maximal flow from source to sink

**Solution:** Firstly, redraw the network, taking initial flow as zero for all edges.

Find an augmenting path from source to sink. Let the path be S-A-B-T. The residual capacities of edges are 7,5 and 8. The bottleneck capacity of this path is 5. Hence update the flow on this path by 5 units. Now the new flow is as follows:

Similarly by considering augmenting path S-D-C-T, the bottleneck capacity is 2. Hence flow would be updated by 2 along this path.

Next augmenting path would be S-D-A-C-T and its bottleneck capacity is 2. Hence new flow at this path will be increased by 2.

Next possible augmenting path is S-A-C-T, with bottleneck capacity 1. So flow at this path would be incremented by 1.

Now, no more augmenting path left. Hence process terminates here. The maximal flow is 5+5=10

**In- Text Questions**

1. What is basic feasible solution? What are different methods of finding basic feasible solution for a transportation problem?
2. Which method should be used for finding BFS and why?
3. Can transportation problem be solved by LPP?
4. Differentiate between Stepping Stone and MODI Method. Which one is more effective and why?
5. Is there any relationship between transportation problem and assignment problem? If yes, explain
6. What is the basic nature of transportation problem, maximization or minimization? Can both type of problems be solved using transportation problem?
7. True or False?
   (a) To balance a transportation model, it may be necessary to add both a dummy source and a dummy destination.

(b) The amounts shipped to a dummy destination represent surplus at the shipping source.

(c) The amounts shipped from a dummy source represent shortages at the receiving destinations.

8. In each of the following cases, determine whether a dummy source or a dummy destination must be added to balance the model.

(a) Supply: *al* = 10, *a2* = 5, *a3* = 4, *a4* = 6
Demand: bi = 10, *b2* = 5, *b3* = 7, *b4* = 9

(b) Supply: *aJ* = 30, *Q2* = 44
Demand: bJ = 25, *bi* = 30, *b3* = 10

## 1.5 SUMMARY

In this chapter the concept of linear programming problem is explained with the help of real life problems. Some real life situations such as production model, Investment model, cost minimization model, production model, man power scheduling model and paper trim loss problems are formulated using linear programming Problem. The basic essential elements of LPP, assumptions of LPP and conditions under which LPP can be used are explained in detail. In next section transportation problem and assignment problems are explained as distribution models. All the methods of finding Basic Feasible Solution (BFS) are explained in detail. Transportation problem is formulated as LPP. Method of finding optimal solution is explained. Hungarian method is explained for finding the optimal solution of assignment problem. All techniques are illustrated with examples

## 1.6 GLOSSARY

- Linear Programming Problem- The mathematical model of some real life situation consisting of linear objective function, linear set of constraints along with non-negativity condition is called Linear Programming Problem.
- Decision Variable- X1, X2, X3, etc., are used to denote the activities. These are referred to as decision variables.
- Integer Programming Problem- A mathematical model, where decision variables can assume only integral values is called Integer Programming Problem.

- Objective Function- The goal of an LPP (Linear Programming Problem) is mathematically represented by a Linear function using decision variables, this is called objective function.
- Constraints- The set of linear constraints are the set of linear inequalities and/or equalities which is just mathematical expression corresponding to the restrictions imposed on the model such as budget constraint, space constraint, labor constraint.
- Optimization – The word optimization is used for maximization or minimization of objective function.
- Balanced transportation Problem- A transportation problem where total supply = total demand is called balanced transportation problem.
- Basic Feasible Solution (BFS)- Set of solution satisfying all the constraints having some positive values and some zero values is called BFS.

## 1.7 ANSWERES TO IN-TEXT QUESTIONS

**Section 1.3**

Ans1 A Linear Programming model essentially consistsof three components.

v) The linear objective function

vi) The set of linear constraints

vii) Non-negativity of decision variables

Ans2 Min Z= - Max Z

Ans3 Because

4. A large variety of problems in diverse field can be represented as linear programming models.
5. Efficient and simple techniques are available for solving linear programming problems.
6. Data variation can be handled through linear programming models with ease.

Ans4 no. In LPP objective function and set of constraints should be linear.

**Section 1.4**

Ans1 Set of solution satisfying all the constraints having some positive values and some zero values is called basic feasible solution. Different methods are:

4. North- West Corner Rule
5. Least Cost Matrix Method
6. Vogel's Approximation Method (VAM)

Ans2 Vogel's Approximation Method (VAM), because it usually reduces number of iterations for finding optimal solution.

Ans3 yes.

Ans 4 For difference refer the algorithm given in text. MODI is preferred over Stepping Stone, as MODI converges faster as compared to Stepping Stone.

Ans5 yes, Assignment problem is a special case of transportation problem. In this case supply and demand of each source and destination is taken as 1.

Ans6. Minimization. Yes, both type can be solved. For solving maximization , it should be converted into minimization and the should be solved.

Ans7

a) False
b) True
c) True

Ans8

a) Dummy Source with supply of 6 units
b) Dummy Destination with demand 9 units

## 1.8 SELF-ASSESSMENT QUESTIONS

Q1. A company produces two types of dolls, regular doll and premium doll. The sales volume for regular doll is at least 80% of the total sales of both the dolls. However, the company cannot sell more than 100 units of regular dolls per day. Both dolls use one special rubber, of which the maximum daily availability is 240 pounds. Regular doll consumes 2 pounds per unit whereas premium doll uses 4 pounds per unit of this special rubber. Company earns profit of $20 and $50, from regular doll and premium doll respectively. Formulate the given condition as a linear programming problem to determine the optimal product mix for the company.

Q2. Alumco manufactures aluminium sheets and aluminium bars. The maximum production capacity is estimated at either 800 sheets or 600 bars per day. The maximum daily demand is 550 sheets and 580 bars. The profit per ton is $40 per sheet and $35 per bar. Formulate the given condition as a linear programming problem to determine the optimal daily production mix.

Q3. An investor wishes to invest $10,000 over the next year in two types of bonds. Bond *A* yields 7% and bond B yields 11%. Past experiences suggests an allocation of at least 25% in *A* and at most 50% in *B.* Moreover, investment in *A* should be at least half the investment in *B.* Formulate the given condition as a linear programming problem to to help the investor that how should the fund be allocated in two bonds?

Q4. The Standard paper company produces paper rolls with a standard width of 20 m each. Special customer orders with different widths are produced by splitting the standard rolls. The typical order received on one day is summarized as follows:

| Order | Desired Width (m) | Desired number of Rolls |
|---|---|---|
| 1 | 5 | 150 |
| 2 | 7 | 200 |
| 3 | 9 | 300 |

Formulate the above problem as a linear programming problem to meet the order with minimum trim loss.

Q5 Find the optimal transportation schedule for the following transportation problem, where entries in the table are corresponding costs for transporting one unit from plant to respective warehouse. Compare optimal cost when initial BFS was

i. Least Cost Matrix Method
ii. North-West Corner Rule
iii. VAM

| **Plant** | **Warehouse** | | | **Supply** |
|---|---|---|---|---|
| | **W1** | **W2** | **W3** | |
| **A** | 28 | 17 | 26 | 500 |
| **B** | 19 | 12 | 16 | 300 |
| **Demand** | 250 | 250 | 500 | |

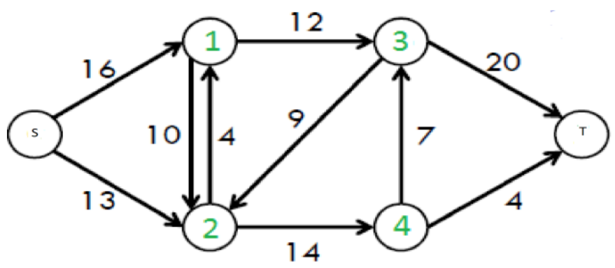
Q6. Crompton has three factories - X, Y, and Z. It supplies its products to four distributors located in different states. The production capacities of these factories are 200, 500 and 300 per month respectively. The Demand of distributors are given in tables. The values written in tables are net returns corresponding to each factory -distributor pair per unit.

| **Factory** | **Distributors** | | | | **Capacity** |
|---|---|---|---|---|---|
| | **A** | **B** | **C** | **D** | |
| **X** | 12 | 18 | 6 | 25 | 200 |
| **Y** | 8 | 7 | 10 | 18 | 500 |
| **Z** | 14 | 3 | 11 | 20 | 300 |
| **Demand** | 180 | 320 | 100 | 400 | |

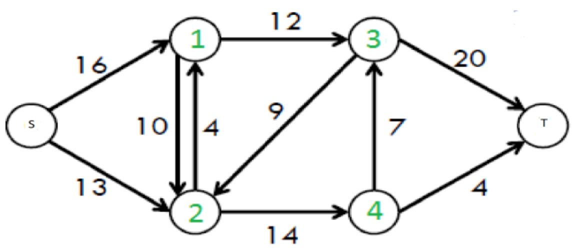
Determine a suitable allocation to maximize the total net return. Find the conditional solution also, i.e. if X can't transport to C and Z can't transport to B.

Q7. What are real life applications of shortest route problem.

Q8. Find the shortest route from source to all other nodes for the following graph.




Q9. Find the maximal flow using Ford-Fulkerson Algorithm.



## 1.9 REFERENCES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- Quantitative Techniques in Management. 5th Edition N.D. Vohra
- Operations Research (Theory Methods & Applications), S.D. Sharma
- Operations Research: Concepts, Problems and Solutions V.K. Kapoor


## 1.10 SUGGESTED BOOKS


- *Operations Research*. An Introduction. Tenth Edition. Hamdy A. *Taha*.
- Quantitative Techniques in Management. 5th Edition N.D. Vohra
- Operations Research (Theory Methods & Applications), S.D. Sharma
- Operations Research: Concepts, Problems and Solutions V.K. Kapoor

**LESSON 2**

# MULTICRITERIA DECISION MODELS

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## STRUCTURE

2.1. Learning Objectives
2.2. Introduction Of Goal Programming (GP)
2.3. Model Formulation \Modeling
2.3.1. Steps Of Goal Programming (GP) Model Formulation
2.3.2. Concept Of Labor Goal
2.3.3. Concept Of Profit Goal
2.3.4. Concept Of Material Goal
2.4. Alternative Forms of Goal Constraints
2.5. Analysis Of Goal Programming (GP) Graphically
2.6. Simplex Method (Modified) Applied to Goal Programming (GP) Problems
2.7. Premptive Goal Programming and Non-Preemptive (Weighted) Goal Programming
2.8. Applications Of Goal Programming (GP) In Management Science
2.9. Summary
2.10. Self-Assessment Exercises
2.11. Objective Questions
2.12. Suggested Readings/References and Glossary

## 2.1 LEARNING OBJECTIVES

After learning this Chapter, you will be able to:

- Explain the concept of Goal Programming (GP).
- Formulate (Modeling) business/industry decision issues of multiple targets

as goal programming (GP) problems.

- Describe the graphical solution used to solve the goal programming (GP) problems.
- Describe the Simplex Method (Modified) solution used to solve the goal programming (GP) problems.

## 2.2 INTRODUCTION OF GOAL PROGRAMMING (GP)

**Goal Programming(GP) is a form of a Linear Programming (LP) that includes several purposes instead of a single objective (also called goal/objective).**

According to Romero (1992), the concept of GP was introduced by Charnes and Cooper (1961). They did not present it as a unique or revolutionary methodology but only as an extension of LP. Ijiri (1965) developed the cocept of different priority levels to the goals and different weights for the goals at the same priority level. Lu (1972) and Ignizio (1976) have discussed the subject of GP in detail and wrote a text book on this subject.

In usual LP models, we examined a single goal that was either maximized or minimized. However, it is seen that a company or an organization frequently has more than one goal, which may include to something other than profit or price.

In fact, a company may have quite a lot of principles, so-called multiple principles, which has been considered in making a judgment as an alternative of just a single goal. For example, in addition to maximizing profit, a company in risk of a labor strike might want to avoid employee layoffs, or a company about to be fined for pollution infractions might want to minimize the emission of pollutants. A company deciding between a number of probable research and improvement projects might want to consider the probability of success of each of the projects, the cost and time required for each, and probable successin making a selection.

In this chapter, we explore the **GP techniques** that can be used to solve problems when they have multiple goals. GP is a deviation of LP in that condition when it considers more than one target in the objective function. The GP models and the LP models are designed in a similar general format with an objective function and linear constraints(also called limits/ restrictions). Also, the solutions of both models (LP model and GP model) are like to be similar.

## 2.3 MODEL FORMULATION \ MODELING

As we know well, the structure of a GP model is alike LP model, with an target function, decision (also so called verdict/ Judgement) variables, and restrictions. Also, the GP models can be solved graphically (like as LP models) when we considered two decision variables in the model.

### 2.3.1. STEPS OF GOAL PROGRAMMING (GP) MODEL FORMULATION

The steps we have taken in the model formulation can be briefly summarized as follows:

1. Define Variables and Constants
2. Formulate Constraints
3. Develop the Objective Function

- **Define Variables and Constants**: The first step of model formation is the definition of probable (choice) variables and the right hand side constants. The right hand side-constants maybe either available resources or specified goal limit value. It requires a careful analysis of the problem in order to identify all significant variables that have some effect on the set of goals (objectives) specified by the decision maker.

- **Formulate Constraints**: Through an analysis of the relationships among choice variables and their relationships to the goal, a set of constraints should be formulated. A constraint maybe either a system constraint that define the relationship between choice variables and the goals. It should be remembered that if there is no deviational variable to minimize in order to achieve in a certain goal, a new constraint must be created. Also, if further refinement of goals and priorities is required , it may be facilitated by decomposing certain deviational variables.

- **Develop the Objective Function**: Through the analysis of the decision marker's goal structure, the objective function must be developed. First, the preemptive priority factors should be assigned to certain deviational variables that are relevant to goal attainment. Second, if necessary differential weights must be assigned to deviational variables at the same priority level. It is imperative that goals at the same priority level be commensurable.

Now, we understand GP modeling by demonstrating through an example with the aim of how to formulate a model. This will be helpful to clarify the main differences between GP and LP.

Now formulating the model as:

$$\begin{aligned}
&Maximize \quad Z = \$40x_1 + 50x_2 \\
&subject\ to \\
&\qquad x_1 + 2x_2 \le 40 \quad hrs.\ of\ labor \\
&\qquad 4x_1 + 3x_2 \le 120 \quad lb.\ of\ clay \\
&\qquad x_1, x_2 \ge 0 \\
&where \\
&x_1 = no.\ of\ bowls\ manufactured \\
&x_2 = no.\ of\ mugs\ manufactured
\end{aligned}$$

Using the *Beaver Creek Pottery Company* example to understand the way a GP model is framed and explain the differences between a LP model and a GP model. However, this model was originally formulated as follows:

- Z is the objective(or goal ) function that represents the total profit to be made from bowls and mugs
- $40 tends to the profit per bowl
- $50 tends to the profit per mug
- The first constraint is for existing labor. It represents a bowl needs 1 hour of labor, a mug needs 2 hours, and total 40 hours of labor are existing daily.
- The second constraint is for clay, and it shows that each bowl needs 4 pounds of clay, each mug needs 3 pounds, and the daily limit of clay is 120 pounds totally.

This is known as a **Standard LP model formulation** because it has a single objective function for return. However, let us suppose that instead of having one objective, the pottery company has several objectives, listed here in order of importance:

i. To ignore layoffs, the company does not need to consume less than 40 hours of labor per day.

ii. The company would like to get an adequate profit limit value of $1600 per day.

iii. Because the clay must be put in storage in a special place so that it does not dry out, the company prefers not to keep more than 120 pounds on hand each day.

iv. Because high overhead cost result when the plant is kept open past normal hours, the company would like to minimize the amount of overtime.

These several aims are stated to as goals in the perspective of the GP technique. The company would, naturally, like to come as close as possible to attaining each of these targets. Because the usual form of the LP model considers only one objective, we must create an alternative form of the model to reproduce these multiple goals.

Now, the first step in formulating a GP model is to convert the LP model constraints into objectives (goals).

**The different aims in a GP problem are denoted to as goals (objectives).**

### 2.3.2. CONCEPT OF LABOR GOAL

The first goal of the pottery company is to ignore underutilization of labor that is, using less than 40 hours of labor each day. To denote the probability of underutilizing labor, the LP constraint for labor, $x_1 + 2x_2 \le 40$ hours of labor, is rewriting as

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40 \quad hrs.$$

This reformulated expression is stated as a goal constraint.

Here, the two new variables, $d_1^-$ and $d_1^+$, are so-called **deviational variables**. They denote the number of labor hours fewer than 40 ($d_1^-$) and the number of labor hours more than 40 ($d_1^+$). More precisely, $d_1^-$ denotes labor underutilization, and $d_1^+$ denotes overtime.

For instance, suppose $x_1 = 5$ bowls and $x_2 = 10$ mugs, then a total of 25 hours of labor have been consumed. Replacing these values into our goal constraint then it gives

$$\Rightarrow (5) + 2(10) + d_1^- - d_1^+ = 40$$

$$\Rightarrow \quad 25 + d_1^- - d_1^+ = 40$$

**All goal constraints are similar that contain deviational variables.**

Because only 25 hours were used in manufacture, labor was underutilized by 15 hours (40 - 25 = 15). Thus, if we suppose $d_1^-$ = 15 hours and $d_1^+$ = 0 (because no overtime is exists there), then we have

$$\Rightarrow 25 + d_1^- - d_1^+ = 40$$
$$\Rightarrow \quad 25 + 15 - 0 = 40$$
$$\Rightarrow \quad 40 = 40$$

**A positive deviational variable $d_1^+$ is the quantity through which a target level is overdone.**

**A negative deviational variable $d_1^-$ is the quantity through which a target level is underperformed.**

Now consider the case:

where $x_1 = 10$ bowls

And $x_2 = 20$ mugs

This indicates that a total of 50 hours have been used for manufacture, or 10 hours above the target level of 40 hours. This additional 10 hours is overtime.

Thus, $d_1 = 0$ (because there is no underutilization)

And $d_1^+ = 10$ hours

In each of these two brief examples, at least one of the deviational variables equated zero. As in the first example we can see $d_1^+ = 0$, and in the second example we can see, $d_1 = 0$.

This is because, it is difficult to use fewer than 40 hours of labor and more than 40 hours of labor at the same time duration. Of course, both deviational variables, $d_1$ and $d_1^+$, could have equated zero, if exactly 40 hours were used in manufacture. These examples explain one of the important features of GP that can be stated as

**In a Goal Constraint, at least one or both of the deviational must equal zero.**

The next step in formulation, our GP model is to get the goal of not using less than 40 hours of labor. We do this by creating a new setup of objective function:

$$Minimize \quad P_1 d_1^-$$

As we know well that, the objective function in all GP models is to minimize deviation from the goal constraint levels. In this objective function, the goal is to minimize $d_1^-$, the underutilization of labor. If $d_1^-$ = 0, then we would not be using less than 40 hours of labor. Thus, it is our aim to make $d_1^-$ equal zero or the minimum quantity possible.

The symbol $P_1$ in the objective function describes the minimization of $d_1^-$ as the first-priority goal. This indicates that the first step will be to minimize the value of $d_1^-$ before any other goal is introduced,when this model is resolved.

**In a GP model, the objective function pursues to minimize the deviation from targets in order of the goal priorities.**

Also, the fourth goal-priority in this issue is related to the labor constraint. The fourth goal which is denoted by $P_4$ and considered to minimize overtime. Remember that hours of overtime are denoted by $d_1^+$;

Therefore, the objective function is given by

$$Minimize \quad P_1 d_1^-, P_4 d_1^+$$

As earlier, the objective is to minimize the deviational variable $d_1^+$. Further, if $d_1^+ = 0$, there would be no overtime throughly. In the calculation of this model, the solution to this fourth-level goal will not be attempted until goals one, two, and three have been solved.

### 2.3.3. CONCEPT OF PROFIT GOAL

In our GP model, the second goal is to obtain a daily profit of $1,600. Remember that the original LP objective function was defined as

$$Z = 40x_1 + 50x_2$$

Now we redefines this objective function as a goal constraint that follows the target level such as

$$40x_1 + 50x_2 + d_2^- - d_2^+ = \$1600$$

The deviational variables $d_2^-$ and $d_2^+$ denote the amount of profit less than $1,600 ($d_2^-$) and the amount of profit exceeding $1,600 ($d_2^+$), respectively. The pottery company's goal to reaching $1,600 in profit is symbolized in the objective function as

$$\textit{Minimize} \quad P_1 d_1^-, P_2 d_2^-, P_4 d_1^+$$

Here, It is seen that only $d_2^-$ is being minimized, not $d_2^+$, since it is reasonable to accept that the pottery company would be agreeable to all profits in additional of $1,600 (i.e., it does not need to minimize $d_2^+$, additional profit). At the second-priority level by minimizing $d_2^-$, the pottery company expectations that $d_2^-$ will equal zero, which will outcome in at least $1,600 in profit.

### 2.3.4. CONCEPT OF MATERIAL GOAL

The third goal of the company is to ignore more than 120 pounds of clay on hand every day. The goal constraint is

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120 \quad lb.$$

Since the deviational variable $d_3^-$ denotes the amount of clay fewer than 120 pounds, and $d_3^+$ denotes the amount in additional of 120 pounds, this goal can be reproduced in the objective function such as

$$\textit{Minimize} \quad P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^+$$

The term $P_3 d_3^+$ indicates the company's requirements to minimize $d_3^+$, the quantity of clay in addition of 120 pounds. The $P_3$ term indicates third most important goal of the pottery company.

The whole GP model can now be defined symbolically as follows:

$$\text{Minimize} \quad P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+$$
$$\text{subject to}$$
$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$
$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1600$$
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$
$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

The simple difference between this model and the standard LP model is that the objective function terms are not summed to equal a total value, Z. The reason behind this is that, the deviational variables in the objective function denotes different unit of measure. For instance, $d_1^-$ and $d_1^+$ indicates hours of labor, $d_2^-$ indicates dollars, and $d_3^+$ indicates pounds of clay. It would be irrelevant to sum hours, dollars, and pounds. The objective function in a GP model specifies only that the deviations from the goals represented in the objective function be minimized individually, in order of their priority.

**Since the deviational variables often have different units of measure then the terms are not summed in the objective function logically.**

## 2.4 ALTERNATIVE FORMS OF GOAL CONSTRAINTS

Now, suppose we want to modify the prior GP model so that our fourth-priority goal boundaries overtime to 10 hours in place of minimizing overtime. Remember that the goal constraint for labor is given by

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$

$d_1^+$ denotes overtime in this goal constraint. Since the new fourth-priority goal is to bound overtime to 10 hours, the goal constraint is developed as follows:

$$d_1^+ + d_4^- + d_4^+ = 10$$

Even though this goal constraint seems unfamiliar, it is satisfactory in GP to have an expression with all deviational variables. In this expression, $d_4^-$ denotes the quantity of overtime fewer than 10 hours, and $d_4^+$ denotes the quantity of overtime more than 10 hours.

Since, the company requirements to bound overtime to 10 hours, $d_4^+$ is minimized in the objective function:

$$Minimize \quad P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+$$

| **All deviational variables can include in Goal constraints.** |
|---|

Next, consider the inclusion of a fifth-priority goal to this example. Assume that the pottery company has limited warehouse space, so it can manufacture no more than 30 bowls and 20 mugs daily. If probable, the company would like to manufacture these amounts. However, because the profit for mugs is more than the profit for bowls (i.e., $50 rather than $40), it is more significant to reach the goal for mugs. This fifth goal necessitates that two new goal constraints be formed, as follows:

$$x_1 + d_5^- = 30 \quad bowls$$
$$x_2 + d_6^- = 20 \quad mugs$$

Here, It is notice that the positive deviational variables $d_5^+$ and $d_6^+$ have been removed from these goal constraints. The reason behind to do this as the statement of the fifth goal specifies that "*no more than 30 bowls and 20 mugs*" can be produced. Further, positive deviation, or over production, is not possible.

Since, the genuine goal of the company is to reach the levels of manufacturing shown in these two goal constraints, the negative deviational variables $d_5^-$ and $d_6^-$ are minimized in the objective function. However, remember that it is more significant to the company to reach the goal for mugs because mugs make more profit. This situation is reflected in the objective function, such as:

$$Minimize \quad P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, 4P_5d_5^- + 5P_5d_6^-$$

Since, the goal for mugs is more significant instead of the goal for bowls, the level of significance should be in ratio to the quantity of profit (i.e., $50 for each mug and $40 for each bowl). Therefore, the goal for mugs is more significant than the goal for bowls in the percentage of 5 to 4.

Here, the coefficient 5 and 4 are referred to as weights for $P_5d_6^-$ and $P_5d_5^-$, respectively. Thus, at the fifth priority level, the minimization of $d_6^-$ is "weighted" greater than the minimization of $d_5^-$. When this model is resolved, the attainment of the goal for minimizing $d_6^-$ (bowls) is more significant, although both goals are at the equal priority level.

**At the same priority level, two or more goals can be assigned weights to specify their relative significance.**

Here it is notice that, these two weighted goals have been summed due to both are at the same priority level. At this individual priority level, their sum characterises achievement of the desired goal. The whole GP model, with the new goals for both overtime and production, is developed as:

$$\text{Minimize} \quad P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, 4P_5d_5^- + 5P_5d_6^-$$

*subject to*

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$
$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1600$$
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$
$$d_1^+ + d_4^- + d_4^+ = 10$$
$$x_1 + d_5^- = 30$$
$$x_2 + d_6^- = 20$$
$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \geq 0$$

## 2.5 ANALYSIS OF GOAL PROGRAMMING (GP) GRAPHICALLY

Only those linear GP problems which involve two decision variables can be solved by the graphical method. This method is quiet similar to the graphical method of LP. The graphical method is used in LP to maximize the objective function with one goal only, whereas in GP, it is used to minimize the deviation from a set of multiple goals. Here the deviation from the goal of highest priority are minimize as much as possible and then the deviations in the other goals in order of priority are minimized so that the achievements of the goals of higher order are not affected.

Following procedural steps are employed in the process, after the problem has been formulated.

**Step-1:** Plot all structural constratints and identify the feasible region. In case, no structural constraints exists, the feasible region is that area where both $x_1$ and $x_2$

are $\geq 0$ (the non-negative quadrant).

**Step-2:** Plot the lines corresponding to the goal constraints. To accomplish this, set the deviational variables in the goal constraint equal to zero and plot the resulting equation.

**Step-3:** Identify the top-priority solution. For this determine the point or points within the feasible region that satisfy the highest priority goal.

**Step-4:** Move to the goal which has the next highest priority and determine the " best" solution(s) already achieved for goals of highest priority.

**Step-5:** Repeat step 4 untill all priority level have been investigated.

For clear understanding the method is illustrated with the help of following example.

The original GP model for Beaver Creek Pottery Company, defined at the beginning of this chapter, will be used as

an example:

$$\begin{aligned}
&\textit{Minimize} \quad P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+ \\
&\textit{subject to} \\
&x_1 + 2x_2 + d_1^- - d_1^+ = 40 \\
&40x_1 + 50x_2 + d_2^- - d_2^+ = 1600 \\
&4x_1 + 3x_2 + d_3^- - d_3^+ = 120 \\
&x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0
\end{aligned}$$

To graph this model, the deviational variables in each goal constraint are set equal to zero, and we graph each subsequent equation on a set of coordinates. Here, Figure-1 is a graph of the three goal constraints for this model.

Notice that in Figure-1, there is no feasible solution space indicated, as in a regular LP model. This is because all three goal constraints are equations; thus, all solution points are on the constraint lines.

The solution logic in a GP model is to try to attain the goals in the objective function, in order of their priorities. As a goal is achieved, the next highest-ranked goal is then considered. However, a higher-ranked goal that has been achieved is never given up in order to achieve a lower-ranked goal.

**Graphical solution illustrates the GP solution logic-seeking to achieve goals by minimizing deviation in order of their priority.**

In this example we first consider **the first-priority goal is minimizing** $d_1^-$. The relationship of $d_1^-$ and $d_1^+$ to the goal constraint is shown in Figure-2. The area below the goal constraint line $x_1 + 2x_2 = 40$ represents possible values for $d_1^-$, and the area above the line represents values for $d_1^+$. In order to achieve the goal of minimizing $d_1^-$, the area below the constraint line corresponding to $d_1^-$ is eliminated, leaving the shaded area as a possible solution area.

Next, we consider **the second-priority goal is minimizing** $d_2^-$. In Figure-3, the area below the constraint line 40x1 + 50x2 = 1,600 represents the values for $d_2^-$, and the area above the line represents the values for $d_2^+$. To minimize $d_2^-$, the area below the constraint line corresponding to $d_2^-$ is eliminated. Notice that by eliminating the area for $d_2^-$, we do not affect the first-priority goal of minimizing $d_1^-$.

**One goal is never achieved at the expense of another higher-priority goal.**

Next, we consider **the third-priority goal is minimizing** $d_3^+$. Figure-4 shows the areas corresponding to $d_3^-$ and $d_3^+$. To minimize $d_3^+$, the area above the constraint line $4x_1 + 3x_2 = 120$ is eliminated. After considering the first three goals, we are left with the area between the line segments AC and BC, which contains possible solution points that satisfy the first three goals.

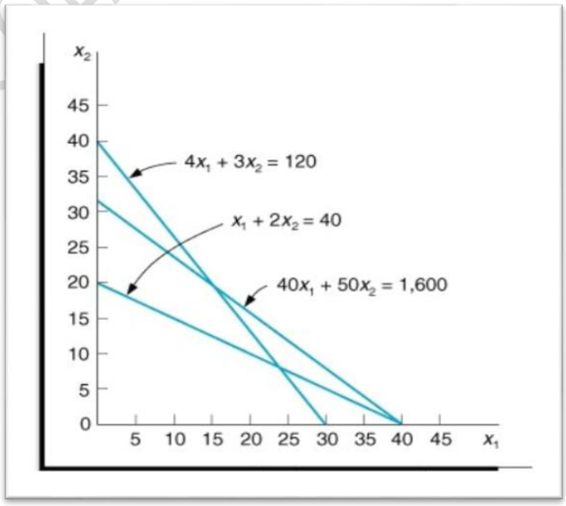



**Figure-1: Graph of Goal Constraints**

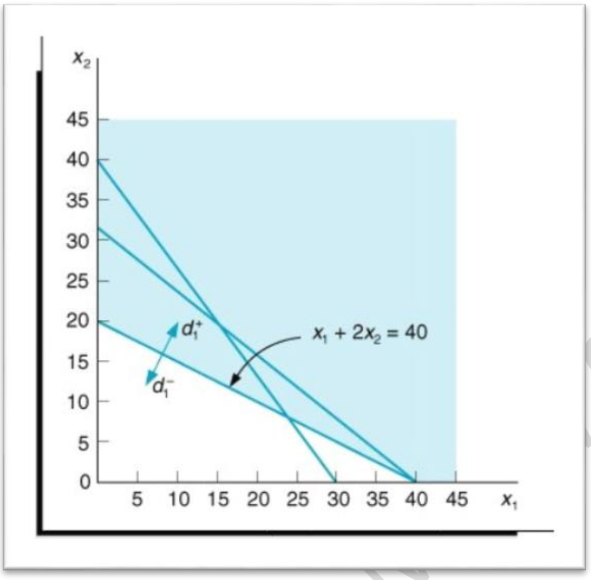



**Figure-2: Minimize $d_1^-$ (The First- Priority Goal)**

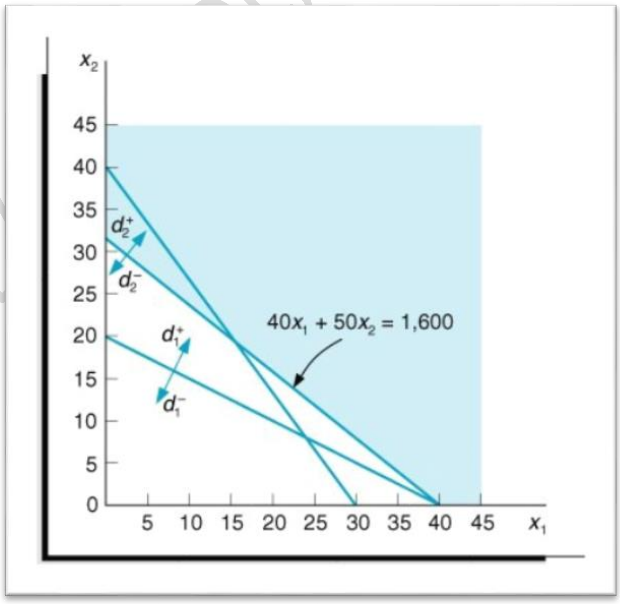



**Figure-3: Minimize $d_2^-$ (The Second- Priority Goal)**

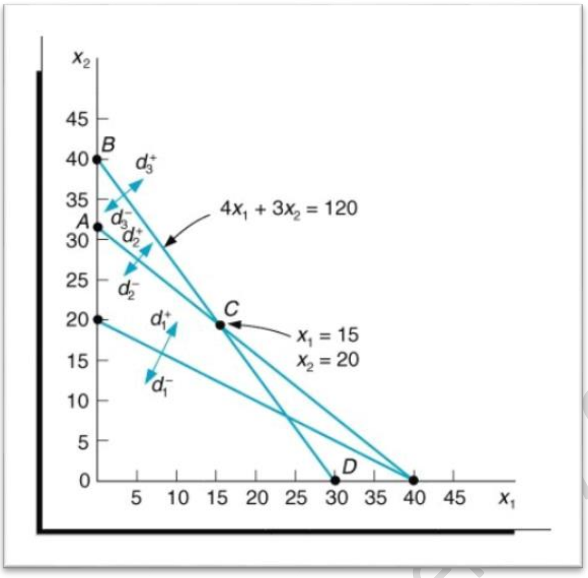



**Figure-4: Minimize $d_3^+$ (The Third- Priority Goal)**

Finally, we must consider **the fourth-priority goal is minimizing $d_1^+$**. To achieve this final goal, the area above the constraint line $x_1 + 2x_2 = 40$ must be eliminated. However, if we eliminate this area, then both $d_2^-$ and $d_3^-$ must take on values. In other words, we cannot minimize $d_1^+$ totally without violating the first- and second-priority goals. Therefore, we want to find a solution point that satisfies the first three goals but achieves as much of the fourth-priority goal as possible.

Point C in Figure-5 is a solution that satisfies these conditions. Notice that if we move down the goal constraint line $4x_1 + 3x_2 = 120$ toward point D, $d_1^+$ is further minimized; however, $d_2^-$ takes on a value as we move past point C. Thus, the minimization of $d_1^+$ would be accomplished only at the expense of a higher-ranked goal.

The solution at point C is determined by simultaneously solving the two equations that intersect at this point. Doing so results in the following solution:

$$x_1 = 50 \quad bowls$$
$$x_2 = 20 \quad mugs$$
$$d_1^+ = 15 \quad hours$$

Because the deviational variables $d_1^-$, $d_2^-$, and $d_3^+$ all equal zero, they have been minimized, and the first three goals have been achieved. Because $d_1^+$ = 15 hours of overtime, the fourth-priority goal has not been achieved. The solution to a GP model such as this one is referred to as the most adequate solution rather than the optimal solution because it fulfils the definite goals as well as possible.

**Further, GP solutions do not always find all goals, and they are not ideal; however, they attain the most suitable solution as well as possible.**

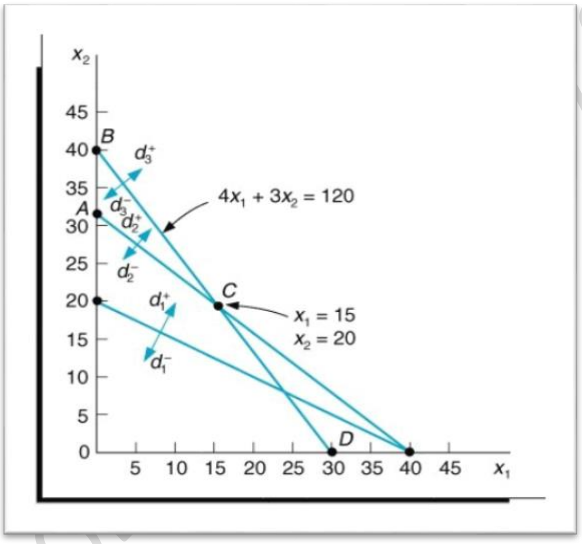



**Figure-5: Minimize $d_1^+$ (The Fourth-Priority Goal) and Final Solution**

## 2.6 SIMPLEX METHOD (MODIFIED) APPLIED TO GOAL PROGRAMMING (GP) PROBLEMS

The simplex method for solving a GP problem is similar to that for a LP problem in the modified form. In this section we shall demonstrate how the algorithm can be modified to solve a GP model. The method of solution of GP problem by modified simplex method, is as follows:

**Step-1. *Formulation of Initial Table*:** Construct the initial simplex table in the same way as for LP problems with the coefficients of the associated variables (decision variables and the deviational variables) placed in the appropriate columns. Now put a thick horizontal line below these entries and write the pre-emptive priority goals $P_1, P_2, \ldots\ldots$, in $x_B$ column, starting from the bottom to the top *i.e.*, first (top) priority $P_1$ is written at the bottom and the least priority is written at the top.

**Step-2.** In GP problem there is no profit maximization or cost minimization in the objective function. Here we minimize the unattained portions of the goal as much as possible, by minimizing the deviational variables through the use of certain pre-emptive priority factors and different weights attached with the deviational derivatives in the objective function. So the pre-emptive priority factor with weight attached with the deviational derivatives in the objective function Z will represent $c_j$ values. Write the $c_j$ -row at the top of the table.

**Step-3. *Test of Optimality*:** Compute the values of $Z_j$ and $c_j - Z_j$ separately for each of the ranked goals $P_1, P_2, \ldots\ldots$. It is because the different goals are measured in different units. $Z_j$ and $c_j - Z_j$ are computed in the same manner as in the usual simplex method of LP problems.

Thus, $c_j - Z_j = c_j - (c_B \; column)^T . (j^{th} \; column)$

and $Z = (c_B \; column)^T . (x_B \; column)$

The optimality criterion $Z_j$ or $c_j - Z_j$ becomes a matrix of size $k \times n$, where $k$ represents the number of pre-emptive priority levels and $n$ is the number of variables including both decision and deviational variables.

**Optimality Test:** Check the $c_1 - Z_1$ row ( $j = 1$ for the top priority $P_1$ ). The top priority goal $P_1$ is said to be achieved if all $c_j - Z_j \geq 0$ in the $P_1$ row or there is zero in $P_1$ row in $x_B$ column.

If atleast one of these entries in $P_1$ row is negative and there is no zero in $P_1$ row in $x_B$ column, then this goal $P_1$ is not achieved and can be improved further, in this case proceed to the next step.

**Step-4. *To find the Entering Vector (or Variable)*:** the variable in the column corresponding to the largest negative $c_1 - Z_1$ value (smallest element) in the $P_1$ row is selected as the entering variable (or vector). In case of tie, check the next lower priority level. The column corresponding to the smallest element (largest negative element) in the lower priority row, out of the columns in which there is a tie in $c_1 - Z_1$ row, is selected as key column (*i.e.*, incoming variable or vector).

**To Find the Outgoing Vector (Or Variable) :** The outgoing vector is selected as in usual simplex method in LP problems. The variable in the row (known as key row), which corresponds to the minimum non-negative value, obtained by dividing the values in the $x_B$ column by the corresponding positive elements (or values) in the key column, is taken as the outgoing variable (or vector).

The element at the intersection of the key row and key column is called key-element.

**Step-5.** As in usual Simplex Method we reduce the key element equal to 1 and with its help all other elements in the key column are reduced to zero. Thus, a new reduced matrix is obtained.

For this matrix again find the values of $Z_j$ or $c_j - Z_j$ for each of the ranked goals $P_1, P_2, .......$ Now again we check $c_1 - Z_1$ row for optimality. If all entries in this $P_1$ row are positive then the goal is achieved (Note that in this situation the values in $x_B$ column, in $P_1$ row will be zero to show that this goal is fully achieved).

If atleast one entry in $P_1$ row is negative then goal $P_1$ is still not achieved. In this case again repeat step- 4 and 5.

**Step-6.** If goal $P_1$ is achieved, then proveed to achieve the next priority goal $P_2$ in the above manner. The goal $P_2$ cannot be improved (achieved) further from the present level if there is positive entry in row $P_1$ (higher priority goal) below the most negative entry in row $P_2$.

Continue this process until the lower priority goal (say $P_i$) is also achieved fully or to the nearest satisfaction. The goal $P_i$ cannotbe improved (achieved) further from the level there is positive entry in higher priority goals $P_1, P_2, .......$ rows below the most negative entry in row $P_i$.

For clear understanding of the above method see the following illustrative example.

**Example:** A company manufactured two products radios and transisters which must be processed through assembly and finishing departments. Assembly has 90 hours available, finishing can handle upto 72 hours of work. Manufacturing one radio requires 6 hours in assembly and 3 hours in finishing. The profit is Rs. 120 per radio and Rs. 90 per transistor. The company has established the following goals and has assigned them priorities $P_1, P_2, P_3$ (where $P_1$ is most important) as follows:

| *Priority* | *Goal* |
|---|---|
| $P_1$ | *Produce to meet a radio goal of 13* |
| $P_2$ | *Reach a profit goal of Rs. 1950* |
| $P_3$ | *Produce to meet a transistor goal of* 5. |

Formulate the problem as a GP problem and find the optimum solution.

**Solution:** *Formulation of the GP problem:* Firstly, the given informations can be put in a tabular form as follows:

| | **Radio $x_1$** | **Transistor $x_2$** | **Time Available In hours** |
|---|---|---|---|
| **Assembly time in hours** | 6 | 3 | 90 |
| **Finishing time in hours** | 3 | 6 | 72 |
| **Profit in Rs.** | 120 | 90 | |

Let $x_1$ and $x_2$ be the numbers of radios and transistors manufactured, respectively.

Also let

$d_1^- \;=\;$ *amount by which the profit goal is underachieved*,

$d_1^+ \;=\;$ *amount by which the profit goal is overachieved*,

$d_2^- \;=\;$ *amount by which the radio goal is underachieved*,

$d_2^+ \;=\;$ *amount by which the radio goal is overachieved*,

$d_3^- \;=\;$ *amount by which the transistor goal is underachieved*,

$d_3^+ \;=\;$ *amount by which the transistor goal is overachieved*.

Then the given problem formulated as a GP problem is as follows:

$$\begin{aligned}
&\text{Minimize } Z = P_1d_2^- + P_2d_1^- + P_3d_3^- \quad \text{i.e.,} \quad \text{Minimize } Z = (d_2^-, d_1^-, d_3^-)\\
&\text{subject to } \; 120x_1 + 90x_2 + d_1^- - d_1^+ = 1950 \quad (\text{Profit goal})\\
&\qquad x_1 + d_2^- - d_2^+ = 13 \quad (\text{Radio goal})\\
&\qquad x_2 + d_3^- - d_3^+ = 5 \quad (\text{Transistor goal})\\
&\qquad 6x_1 + 3x_2 \le 90 \quad (\text{Assembly constraint})\\
&\qquad 3x_1 + 6x_2 \le 72 \quad (\text{Finishing constraint})\\
&\text{and } \; x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \ge 0.
\end{aligned}$$

***Solution of the GP Problem:*** Introducing the slack variables $x_3, x_4$, the above GP problem can be written as follows:

$$\begin{aligned}
&\text{Minimize } Z = P_1d_2^- + P_2d_1^- + P_3d_3^-\\
&\text{subject to } \; 120x_1 + 90x_2 + d_1^- - d_1^+ = 1950\\
&\qquad x_1 + d_2^- - d_2^+ = 13\\
&\qquad x_2 + d_3^- - d_3^+ = 5\\
&\qquad 6x_1 + 3x_2 + x_3 = 90\\
&\qquad 3x_1 + 6x_2 + x_4 = 72\\
&\text{and } \; x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \ge 0.
\end{aligned}$$

Now we shall solve this problem step by step.

Taking $x_1 = 0 = x_2 = d_1^+ = d_2^+ = d_3^+$, we get $d_1^- = 1950$, $d_2^- = 13$, $d_3^- = 5$, $x_3 = 90$, $x_4 = 72$, which is the basic feasible solution.

**Step-1.** ***Formulation of the initial table*:** Now we formulate the starting (initial) table as follows. (As explained in step-1 of 2.6).

| Type | B | $c_B$ | $c_j$ | **0** | **0** | **0** | **0** | $P_2$ | **0** | $P_1$ | **0** | $P_3$ | **0** | **Mini Ratio** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $x_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $d_1^-$ | $d_1^+$ | $d_2^-$ | $d_2^+$ | $d_3^-$ | $d_3^+$ | $x_B / x_1$ |
| | $d_1^-$ | $P_2$ | 1950 | 120 | 90 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1950/120 |
| | $d_2^-$ | $P_1$ | 13 | [1] | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 13/1 (Min) → |
| | $d_3^-$ | $P_3$ | 5 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | — |
| | $x_3$ | 0 | 90 | 6 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 90/6 |
| | $x_4$ | 0 | 72 | 3 | 6 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 72/3 |
| | $c_j - Z_j$ | $P_3$ | 5 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | |
| | | $P_2$ | 1950 | -120 | -90 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | |
| | | $P_1$ | 13 | -1 ↑ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | |

**Step-2.** The $c_j$ row is written at the top of the table.

**Step-3.** ***Test of optimality:*** Here we compute $c_j - Z_j, \quad j = 1, 2, \ldots, 10$.

$$c_1 - Z_1 = 0 - (P_2, P_1, P_3, 0, 0).(120, 1, 0, 6, 3)^T = -120P_2 - P_1 + 0.P_3$$
$$c_2 - Z_2 = 0 - (P_2, P_1, P_3, 0, 0).(90, 0, 1, 3, 6)^T = -90P_2 + 0.P_1 + 1.P_3$$
$$c_3 - Z_3 = 0 - (P_2, P_1, P_3, 0, 0).(0, 0, 0, 1, 0)^T = 0.P_2 + 0.P_1 + 0.P_3$$
$$c_4 - Z_4 = 0 - (P_2, P_1, P_3, 0, 0).(0, 0, 0, 0, 1)^T = 0.P_2 + 0.P_1 + 0.P_3$$
$$c_5 - Z_5 = P_2 - (P_2, P_1, P_3, 0, 0).(1, 0, 0, 0, 0)^T = 0.P_2 + 0.P_1 + 0.P_3$$
$$c_6 - Z_6 = 0 - (P_2, P_1, P_3, 0, 0).(-1, 0, 0, 0, 0)^T = P_2 + 0.P_1 + 0.P_3$$
$$c_7 - Z_7 = P_1 - (P_2, P_1, P_3, 0, 0).(0, 1, 0, 0, 0)^T = 0.P_2 + 0.P_1 + 0.P_3$$
$$c_8 - Z_8 = 0 - (P_2, P_1, P_3, 0, 0).(0, -1, 0, 0, 1)^T = 0.P_2 + 1.P_1 + 0.P_3$$
$$c_9 - Z_9 = P_3 - (P_2, P_1, P_3, 0, 0).(0, 0, 1, 0, 0)^T = 0.P_2 + 0.P_1 + 0.P_3$$
$$c_{10} - Z_{10} = 0 - (P_2, P_1, P_3, 0, 0).(0, 0, -1, 0, 0)^T = 0.P_2 + 0.P_1 + 1.P_3$$

Note that all enteries in the columns corresponding to vectors in the basis are zero. So we may compute $c_j - Z_j$ for columns corresponding to non-basic variables only. The entries in columns corresponding to basic variables will be zero.

$$Z = c_B x_B = (P_2, P_1, P_3, 0, 0).(1950, 13, 5, 90, 72)^T = 1950P_2 + 13P_1 + 5P_3$$

All these enteries are made in the above table.

**Step-4.** The most negative (– ve) entry in $P_1$ (Top priority) row is -1 in Ist column.

$\therefore x_1$ is the incoming variable and by minimum ratio rule $d_2^-$ is the outgoing variable. Thus, the key element is 1 $(= a_{21})$.

**Step-5.** Here, reducing all other elements in key column $c_1$ equal to zero with the help of key element, the next table is as follows:

| **Type** | **B** | $c_B$ | $c_j$ / $x_B$ | **0** $x_1$ | **0** $x_2$ | **0** $x_3$ | **0** $x_4$ | $P_2$ $d_1^-$ | **0** $d_1^+$ | $P_1$ $d_2^-$ | **0** $d_2^+$ | $P_3$ $d_3^-$ | **0** $d_3^+$ | **Mini Ratio** $x_B / d_2^+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d_1^-$ | $P_2$ | 390 | 0 | 90 | 0 | 0 | 1 | -1 | -120 | 120 | 0 | 0 | 390/120 |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $x_1$ | 0 | 13 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | — |
| | $d_3^-$ | $P_3$ | 5 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | — |
| | $x_3$ | 0 | 12 | 0 | 3 | 1 | 0 | 0 | 0 | -6 | **[6]** | 0 | 0 | 12/6 (Min) → |
| | $x_4$ | 0 | 33 | 0 | 6 | 0 | 1 | 0 | 0 | -3 | 3 | 0 | 0 | 33/3 |
| | $c_j - Z_j$ | $P_3$ | 5 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | |
| | | $P_2$ | 390 | 0 | -90 | 0 | 0 | 0 | 1 | 120 | -120 | 0 | 0 | |
| | | $P_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 ↑ | 0 | 0 | |

Here, we again $c_j - Z_j$ compute for columns corresponding to non-basic variables only. All enteries in the columns corresponding to basic variables will be zero.

$$c_2 - Z_2 = 0 - (P_2, 0, P_3, 0, 0).(90, 0, 1, 3, 6)^T = -90P_2 - P_3$$
$$c_6 - Z_6 = 0 - (P_2, 0, P_3, 0, 0).(-1, 0, 0, 0, 0)^T = P_2$$
$$c_7 - Z_7 = P_1 - (P_2, 0, P_3, 0, 0).(-120, -1, 0, -6, -3)^T = P_1 + 120P_2$$
$$c_8 - Z_8 = 0 - (P_2, 0, P_3, 0, 0).(120, -1, 0, 6, 3)^T = -120P_2$$
$$c_{10} - Z_{10} = 0 - (P_2, 0, P_3, 0, 0).(0, 0, -1, 0, 0)^T = P_3$$

The value of $c_j - Z_j$ in $P_1, P_2, P_3$ rows may also be found easily as follows making 1 at the place of key element use it to reduce all enteries in $P_1, P_2, P_3$ rows corresponding to the column of key element to zero.

$$Z = c_B x_B = (P_2, 0, P_3, 0, 0).(390, 13, 5, 12, 33)^T = 390P_2 + 5P_3$$

Since all enteries in $P_1$ row are $\geq 0$ , so the priority goal $P_1$ is achieved.

Now we proceed to achieve that goal $P_2$ , without affecting the achievement of top priority goal $P_1$.

**Step-6.** In the $P_2$ row (in above table) most negative value is -120 in column corresponding to variable $d_2^+$. So this variable is taken as the entering variable. Now by minimumratio rule $x_3$ in 4 th-row is the outgoing vector. Thus, $6(=a_{48})$ is the key element. Dividing this 4 th-row by 6, we make 1 (at this place) and with the help of 1 at this place we reduce all other elements in this $d_2^+$ column equal to zero.

Thus, we get the next simplex table as follows:

| **Type** | **B** | $c_B$ | $c_j$ | **0** | **0** | **0** | **0** | $P_2$ | **0** | $P_1$ | **0** | $P_3$ | **0** | **Mini Ratio** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $x_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $d_1^-$ | $d_1^+$ | $d_2^-$ | $d_2^+$ | $d_3^-$ | $d_3^+$ | $x_B / d_2^+$ |
| | $d_1^-$ | $P_2$ | 150 | 0 | 30 | -20 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 150/30 = 5 |
| | $x_1$ | 0 | 15 | 1 | 1/2 | 1/6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 15/(1/2) =30 |
| | $d_3^-$ | $P_3$ | 5 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 5/1 = 5 |
| | $d_2^+$ | 0 | 2 | 0 | [1/2] | 1/6 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 2/(1/2) = 4 (Min) $\rightarrow$ |
| | $x_4$ | 0 | 27 | 0 | 9/2 | -1/2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 27/(9/2) = 6 |
| | $c_j - Z_j$ | $P_3$ | 5 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | |
| | | $P_2$ | 150 | 0 | -30 | 20 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | |
| | | $P_1$ | 0 | 0 | 0 $\uparrow$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | |

Now, we again compute $c_j - Z_j$ for columns corresponding to non-basic variables only. All enteries in the columns corresponding to basic variables will be zero.

$$c_2 - Z_2 = 0 - (P_2, 0, P_3, 0, 0).(30,\ 1/2, 1, 1/2, 9/2)^T = -30P_2 - P_3$$
$$c_3 - Z_3 = 0 - (P_2, 0, P_3, 0, 0).(-20,\ 1/6, 0, 1/6, -1/2)^T = 20P_2$$
$$c_6 - Z_6 = 0 - (P_2, 0, P_3, 0, 0).(-1, 0, 0, 0, 0)^T = P_2$$
$$c_7 - Z_7 = P_1 - (P_2, 0, P_3, 0, 0).(0, 0, 0, -1, 0)^T = P_1$$
$$c_{10} - Z_{10} = 0 - (P_2, 0, P_3, 0, 0).(0, 0, -1, 0, 0)^T = P_3$$

$$Z = c_B x_B = (P_2, 0, P_3, 0, 0).(150, 15, 5, 2, 27)^T = 150P_2 + 5P_3$$

Again in $P_2$ row $c_2 - Z_2$ is negative, so this solution is not optimal from $P_2$ point of view. Now we take variable $x_2$ in second column corresponding to most negative entry in $P_2$ rows as entering variable, by minimum ratio rule $d_2^+$ in IV row is the outgoing variable, so the key element is $1/2 (= a_{42})$ . Dividing IV row by ½, to make 1, the key element, and then reducing all other elements in 2 nd column to 0, we get the following reduced matrix.

| Type | B | $c_B$ | $c_j$ | **0** | **0** | **0** | **0** | $P_2$ | **0** | $P_1$ | **0** | $P_3$ | **0** | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $x_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $d_1^-$ | $d_1^+$ | $d_2^-$ | $d_2^+$ | $d_3^-$ | $d_3^+$ | |
| | $d_1^-$ | $P_2$ | 30 | 0 | 0 | -30 | 0 | 1 | -1 | 60 | -60 | 0 | 0 | |
| | $x_1$ | 0 | 13 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | |
| | $d_3^-$ | $P_3$ | 1 | 0 | 0 | -1/3 | 0 | 0 | 0 | 2 | -2 | 1 | -1 | |
| | $d_2^+$ | 0 | 4 | 0 | 1 | 1/3 | 0 | 0 | 0 | -2 | 2 | 0 | 0 | |
| | $x_4$ | 0 | 9 | 0 | 0 | -2 | 1 | 0 | 0 | 9 | -9 | 0 | 0 | |
| | $c_j - Z_j$ | $P_3$ | 5 | 0 | 0 | 1/3 | 0 | 0 | 0 | -2 | 2 | 0 | 1 | |
| | | $P_2$ | 150 | 0 | 0 | 30 | 0 | 0 | 1 | -60 | 60 | 0 | 0 | |
| | | $P_1$ | 0 | 0 | 0<br>↑ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | |

Again $c_j - Z_j$ compute for columns corresponding to non-basic variables only.

We get

$$c_3 - Z_3 = 0 - (P_2, 0, P_3, 0, 0).(-30, 0, -1/3, 1/3, -2)^T = 30P_2 + (1/3)P_3$$
$$c_6 - Z_6 = 0 - (P_2, 0, P_3, 0, 0).(-1, 0, 0, 0, 0)^T = P_2$$
$$c_7 - Z_7 = P_1 - (P_2, 0, P_3, 0, 0).(60, 1, 2, -2, 9)^T = P_1 - 60P_2 - 2P_3$$
$$c_8 - Z_8 = 0 - (P_2, 0, P_3, 0, 0).(-60, -1, -2, 2, -9)^T = 60P_2 + 2P_3$$
$$c_{10} - Z_{10} = 0 - (P_2, 0, P_3, 0, 0).(0, 0, -1, 0, 0)^T = P_3$$

And $\quad Z = c_B x_B = (P_2, 0, P_3, 0, 0).(150, 15, 5, 2, 27)^T = 150P_2 + 5P_3$

In the above table we note that there is negative entry -60 in $P_2$. But $P_2$ cannot be improved further as there is positive entry below this element in $P_1$ row (top priority row). Similarly, if we move to improve $P_3$, then also it is not possible as there is positive entry in row $P_1$, below the negative entry in row $P_3$.

Thus, $P_2$ and $P_3$ cannot be improved further.

Hence the solution of the above GP problem is

$x_1 = 13, \quad x_2 = 4, \quad d_1^- = 30, \quad d_3^- = 1, \quad d_1^+ = 0 = d_2^- = d_2^+ = d_3^+$ *i.e.*, radios and 4 transistors should be manufactured.

For this solution the first (top) priority goal $P_1$ is fully achieved (13 radios), the second priority goal $P_2$ is missed by Rs. 30 (here $\text{Profit} = 120 \times 13 + 90 \times 4 = \text{Rs.}1920,$ and $\text{Rs.}1950 - 1920 = \text{Rs.}30$ only), and the last priority goal $P_3$ is also missed by 1 transistor (here 5- 4 = 1).

Hence the optimum solution to GP problem is

- $x_1 = 13, \quad x_2 = 4, \quad Minimize \quad Z = (0, 30, 1)$.

## 2.7 PREEMPTIVE GOAL PROGRAMMING AND NON-PREEMPTIVE (WEIGHTED) GOAL PROGRAMMING

This section presents two algorithms for solving GP. Namely as

1. Preemptive Goal Programming (PEGP)

2. Non-Preemptive Goal Programming (NPGP) or Weighted Goal Programming (WGP).

Both methods are based on representing the multiple goals by a single objective function.

The ***Preemptive Procedure*** starts by prioritizing the goals in order of importance. The model then optimizes the goals one at a time in order of priority and in a manner that does not degrade a higher-priority solution.

In the ***Non-Preemptive Procedure***, *the single objective function is the weighted sum of the functions representing the goals of the problem*.

The proposed two methods do not generally produce the same solution. Neither method, however, is superior to the other, because the two techniques entail distinct decision-making preferences.

As we know very well that, In some situations, a decision maker may face multiple objectives, and there may be no point in an LP's feasible region satisfying all objectives. In such a case, *how can the decision maker choose a satisfactory decision*? Goal programming is one technique that can be used in such situations. The following example illustrates the main ideas of goal programming with both methods (Premptive and Non-Preemptive Procedue).

Example: The Leon Bumit Advertising Agency is tryimg to determine a TV Advertising Schedule for Priceler Auto Company. Priceler has three goals such as:

- Goal 1: Its ads should be seen by at least 40 million high-income men (HIM).
- Goal 2: Its ads should be seen by at least 60 million low-income people (LIP).
- Goal 3: Its ads should be seen by at least 35 million high-income women (HIW).

Leon Burnit can purchase two types of ads: those shown during football games and those shown during soap operas. At most, \$600,000 can be spent on ads. The advertising costs and potential audiences of a one- minute ad of each type are shown in below table. Leon Burnit must determine how many football ads and soap opera ads to purchase.

| Ad | HIM | LIP | HIW | Cost |
|---|---|---|---|---|
| **Football Game** | 7 | 10 | 5 | **100,000** |
| **Soap Opera** | **3** | **5** | **4** | **60.000** |

**Table: Millions of Viewers**

If we let

$x1$ = # of minutes of ads shown during football games

$x2$ = # of minutes of ads shown during soap operas

We can write the constraints of the problem as

$$\begin{aligned} 7x_1 + 3x_2 &\geq 40 \\ 10x_1 + 5x_2 &\geq 60 \\ 5x_1 + 4x_2 &\geq 35 \\ 100x_1 + 60x_2 &\leq 600 \\ x_1, \quad x_2 &\geq 0 \end{aligned}$$

From the Figure-1, we find that no point that satisfies the budget constraint meets all three of Priceler's goals. Thus, the problem has no feasible solution. It is impossible to meet all of Priceler's goals, so Burnit might ask Priceler to identify, for each goal, a cost (per-unit short of meeting each goal) that is incurred for failing to meet the goal.

Burnit can now formulate an LP that minimizes the cost incurred in deviating from Priceler's three goals. The trick is to transform each inequality constraint in that represents one of Priceler's goals into an equality constraint. The cost-minimizing solution might under- satisfy or over-satisfy a given goal, so we need to define the following ***deviational variables***:

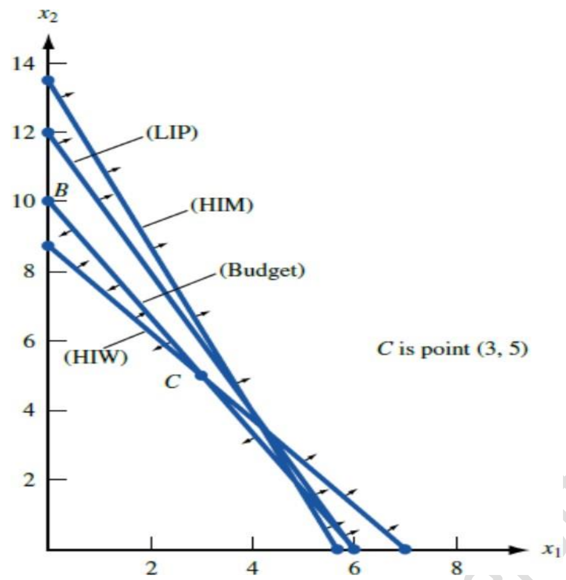



Figure-2.1

$$d_i^+ \;=\; \textit{the amount by which we are over the } i-th \textit{ goal}$$
$$d_i^- \;=\; \textit{the amount by which we are under the } i-th \textit{ goal}$$

Using the deviational variables, we can write

$$7x_1 + 3x_2 + d_1^- - d_1^+ = 40$$
$$10x_1 + 5x_2 + d_2^- - d_2^+ = 60$$
$$5x_1 + 4x_2 + d_3^- - d_3^+ = 35$$

**Non-Preemptive Goal Programming**

Now suppose Priceler determines that

- Each million exposures by which Priceler falls short of the HIM goal costs Priceler a $200,000 penalty because of lost sales.
- Each million exposures by which Priceler falls short of the LIP goal costs Priceler a $100,000 penalty because of lost sales.

- Each million exposures by which Priceler falls short of the HIW goal costs Priceler a $50,000 penalty because of lost sales.

To find the best solution satisfying the above equations, we can write the following model with the objective:

$$\begin{aligned}
&Minimize = 200d_1^- + 100d_2^- + 50d_3^- \\
&7x_1 + 3x_2 + d_1^- - d_1^+ = 40 \\
&10x_1 + 5x_2 + d_2^- - d_2^+ = 60 \\
&5x_1 + 4x_2 + d_3^- - d_3^+ = 35 \\
&100x_1 + 60x_2 + s_4 = 600 \\
&x_1,\ x_2,\ d_i^-,\ d_i^+,\ s_4 \geq 0,\ \forall i
\end{aligned}$$

The optimal solution to the above model is

$$\begin{aligned}
&Z = 250; \quad x_1 = 6, \quad x_2 = 0, \\
&d_1^+ = 2, \quad d_2^+ = d_3^+ = d_1^- = d_2^- = 0 \quad and \quad d_3^- = 5.
\end{aligned}$$

**Preemptive Goal Programming (PGP)**

In our LP formulation of the Burnit example, we assumed that Priceler could exactly determine the relative importance of the three goals. For instance, Priceler determined that the HIM goal was 2 times as important as the LIP goal, and the LIP goal was 2 times as important as HIW goal. In many situations, however, *a decision maker may not be able to determine precisely the relative importance of the goals*. When this is the case, preemptive goal programming may prove to be a useful tool. To apply PGP, the decision maker must rank his or her goals from the most important (goal 1) to least important (goal $n$). The objective function coefficient for the variable representing goal $i$ will be $P_i$ where we assume that

$$P_1 \gg P_2 \gg \dots\dots \gg P_n$$

For the example, we can then write

$$
\begin{aligned}
Minimize &= P_1 d_1^- + P_2 d_2^- + P_3 d_3^- \\
7x_1 + 3x_2 + d_1^- - d_1^+ &= 40 \\
10x_1 + 5x_2 + d_2^- - d_2^+ &= 60 \\
5x_1 + 4x_2 + d_3^- - d_3^+ &= 35 \\
100x_1 + 60x_2 \qquad\qquad + s_4 &= 600 \\
x_1,\quad x_2,\quad d_i^-,\quad d_i^+,\quad s_4 &\geq 0, \quad \forall i
\end{aligned}
$$

Preemptive goal programming problems can be solved by an extension of the simplex known as the goal programming simplex. To prepare a problem for solution by the goal programming simplex, we must compute $n$ Row 0s (objective rows), with the $i$-th row corresponding to goal $i$.

We thus have

$$
\begin{aligned}
&\text{Row 0 - Objective 1 (goal 1)}: Z_1 - P_1 d_1^- = 0 \\
&\text{Row 0 - Objective 2 (goal 2)}: Z_2 - P_2 d_2^- = 0 \\
&\text{Row 0 - Objective 3 (goal 3)}: Z_3 - P_3 d_3^- = 0
\end{aligned}
$$

By organizing these, we have

$$
\begin{aligned}
&\text{Row 0 - Objective 1 (goal 1)}: Z_1 + 7P_1 x_1 + 3P_1 x_2 - P_1 d_1^+ = 40\text{P}_1 \\
&\text{Row 0 - Objective 2 (goal 2)}: Z_2 + 10P_2 x_1 + 5P_2 x_2 - P_2 d_2^+ = 60\text{P}_2 \\
&\text{Row 0 - Objective 3 (goal 3)}: Z_3 + 5P_3 x_1 + 4P_3 x_2 - P_3 d_3^+ = 30\text{P}_3
\end{aligned}
$$

| | **Z** | $x_1$ | $x_2$ | $d_1^+$ | $d_2^+$ | $d_3^+$ | $d_1^-$ | $d_2^-$ | $d_3^-$ | $s_4$ | **RHS** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $Z_1$ | 1 | $7P_1$ | $3P_1$ | $-P_1$ | 0 | 0 | 0 | 0 | 0 | 0 | $40P_1$ |
| $Z_2$ | 1 | $10P_2$ | $5P_2$ | 0 | $-P_2$ | 0 | 0 | 0 | 0 | 0 | $60P_2$ |
| $Z_3$ | 1 | $5P_3$ | $4P_3$ | 0 | 0 | $-P_3$ | 0 | 0 | 0 | 0 | $35P_3$ |
| $d_1^-$ | 0 | 7 | 3 | -1 | 0 | 0 | 1 | 0 | 0 | 0 | 40 |
| $d_2^-$ | 0 | 10 | 5 | 0 | -1 | 0 | 0 | 1 | 0 | 0 | 65 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_3^-$ | 0 | 5 | 4 | 0 | 0 | -1 | 0 | 0 | 1 | | 35 |
| $s_4$ | 0 | 100 | 60 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 600 |

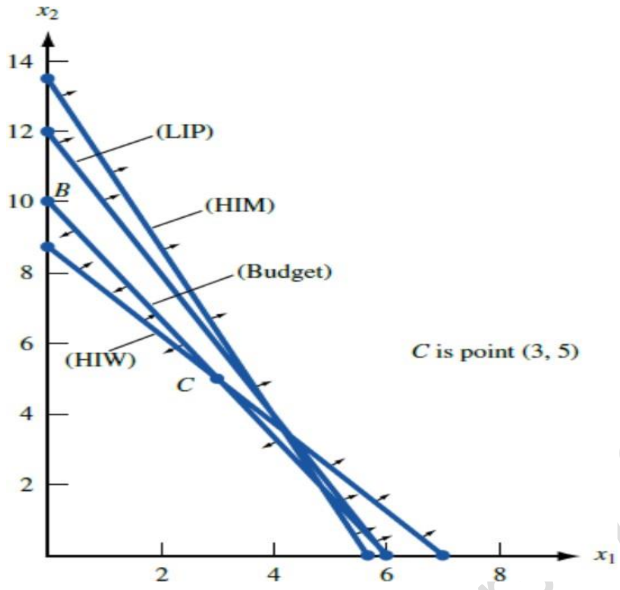
| | Z | $x_1$ | $x_2$ | $d_1^+$ | $d_2^+$ | $d_3^+$ | $d_1^-$ | $d_2^-$ | $d_3^-$ | $s_4$ | RHS |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $Z_1$ | 1 | 0 | 0 | 0 | 0 | 0 | $-P_1$ | 0 | 0 | 0 | 0 |
| $Z_2$ | 1 | 0 | $\frac{5P_2}{7}$ | $\frac{10P_2}{7}$ | $-P_2$ | 0 | $-\frac{10P_2}{7}$ | 0 | 0 | 0 | $\frac{20P_2}{7}$ |
| $Z_3$ | 1 | 0 | $\frac{13P_3}{7}$ | $\frac{5P_3}{7}$ | 0 | $-P_3$ | $-\frac{5P_3}{7}$ | 0 | 0 | 0 | $\frac{45P_3}{7}$ |
| $x_1$ | 0 | 1 | $\frac{3}{7}$ | $\frac{-1}{7}$ | 0 | 0 | $\frac{1}{7}$ | 0 | 0 | 0 | $\frac{40}{7}$ |
| $d_2^-$ | 0 | 0 | $\frac{5}{7}$ | $\frac{10}{7}$ | -1 | 0 | $-\frac{10}{7}$ | 1 | 0 | 0 | $\frac{20}{7}$ |
| $d_3^-$ | 0 | 0 | $\frac{13}{7}$ | $\frac{5}{7}$ | 0 | -1 | $-\frac{5}{7}$ | 0 | 1 | | $\frac{45}{7}$ |
| $s_4$ | 0 | 0 | $\frac{120}{7}$ | $\frac{100}{7}$ | 0 | 0 | $-\frac{100}{7}$ | 0 | 0 | 1 | $\frac{200}{7}$ |

| | Z | $x_1$ | $x_2$ | $d_1^+$ | $d_2^+$ | $d_3^+$ | $d_1^-$ | $d_2^-$ | $d_3^-$ | $s_4$ | RHS |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $Z_1$ | 1 | 0 | 0 | 0 | 0 | 0 | $-P_1$ | 0 | 0 | 0 | 0 |
| $Z_2$ | 1 | 0 | $-P_2$ | 0 | $-P_2$ | 0 | 0 | 0 | 0 | $\frac{-P_2}{10}$ | 0 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $Z_3$ | 1 | 0 | $P_3$ | 0 | 0 | $-P_3$ | 0 | 0 | 0 | $\frac{-P_3}{20}$ | $5P_3$ |
| $x_1$ | 0 | 1 | $\frac{3}{5}$ | 0 | 0 | 0 | $\frac{1}{7}$ | 0 | 0 | $\frac{1}{100}$ | 6 |
| $d_2^-$ | 0 | 0 | -1 | 0 | -1 | 0 | 0 | 1 | 0 | $-\frac{1}{10}$ | 0 |
| $d_3^-$ | 0 | 0 | 1 | 0 | 0 | -1 | 0 | 0 | 1 | $-\frac{1}{20}$ | 5 |
| $d_1^+$ | 0 | 0 | $\frac{6}{5}$ | 1 | 0 | 0 | -1 | 0 | 0 | $\frac{7}{100}$ | 2 |

| **Priorities** | | | | | **Deviational Variables** | | |
|---|---|---|---|---|---|---|---|
| **Highest** | **Medium** | **Lowest** | $x_1$ | $x_2$ | HIM | LIP | HIW |
| **HIM** | LIP | HIW | 6 | 0 | 0 | 0 | 5 |
| **HIM** | HIW | LIP | 5 | $\frac{5}{3}$ | 0 | $\frac{5}{3}$ | $\frac{10}{3}$ |
| **LIP** | HIM | HIW | 6 | 0 | 0 | 0 | 5 |
| **LIP** | HIW | HIM | 6 | 0 | 0 | 0 | 5 |
| **HIW** | HIM | LIP | 3 | 5 | 4 | 5 | 0 |
| **HIW** | LIP | HIM | 3 | 5 | 4 | 5 | 0 |

When a preemptive goal programming problem involves only two decision variables, the optimal solution can be found graphically. For example, suppose HIW is the highest priority goal, LIP is the second-highest, and HIM is the lowest. From the Figure, we find that the set of points satisfying the highest-priority goal (HIW) and the budget constraint is bounded by the triangle ABC.




Among these points, we now try to come as close as we can to satisfying the second-highest-priority goal (LIP). Unfortunately, no point in triangle ABC satisfies the LIP goal. We see from the figure, however, that among all points satisfying the highest-priority goal, point C (C is where the HIW goal is exactly met and the budget constraint is binding) is the unique point that comes the closest to satisfying the LIP goal.

Simultaneously solving the following equations, we find that point C (3, 5) is the solution that satisfies both goals and closest to satisfying the LIP goal.

$$5x_1 + 4x_2 = 35$$
$$100x_1 + 60x_2 = 600$$

We can use computer system i.e, MS Excel Solver to solve preemptive GP models.

## 2.8 APPLICATIONS OF GOAL PROGRAMMING (GP) IN MANAGEMEMENT SCIENCE

GP has a close correspondence with decision making. As managers are constantly called upon to make decisions in order to solve problems, this technique is particularly relevant in the field. Business success relies on effective decision making processes, and GP models can assist. In particular assigned weights can express the intensity with which the goals are strived for. Moreover in management the multiple GP approach can be considered as an extension of the widely used break-even analysis. GP been applied in different management fields, such as accounting like budgeting, cost allocation, corporate social reporting etc., finance like asset management, portfolio selection etc., marketing like sales operation, media planning, operations like inventory management, transportation etc., and natural resources. The increasing popularity of GP and usefulness for decision making policies are particularly evident in some areas, such as portfolio management, marketing and strategic management etc.

## 2.9 SUMMARY

- In this chapter, at the beginning, we explore introduction to GP as a powerful tool to tackle multiple and incompatible goals of any enterprise some of which may be non-economic in nature.
- The next point in discussion has been on the concepts of GP. The differences between GP and LP have been brought out. The distinguishing features of GP revolve around its ability to use the ordinal principle of preemptive priority structure of the goals of management which may be incommensurable.
- The formulations of GP models with its steps have been covered with the typical and comprehensive examples.
- In the graphical method of solving the GP problem, one problem was formulated and solved graphically for a meaningful appreciation.
- The optimal solutions of GP problems by modified simplex method with its steps have been covered with the typical and comprehensive examples.

## 2.10 SELF-ASSESSMENT QUESTIONS

1. What is GP?
2. Identify the major differences between LP and GP.
3. What is GP? State clearly its assumptions.
4. An office equipment manufacturer produces two kinds of products, chairs and lamps. Production of either a chair or a lamp requires 1 hour of production capacity in the plant. The plant has a maximum capacity of 10 hours per week. The gross margin from the chair is Rs. 80 and Rs. 40 for that of a lamp. Formulate the problem as a GP problem if the goal of the firm is to earn a profit of Rs. 800 per week. Formulate the

problem.

**Answer:** *The GP can be stated as:*

$$\text{Minimize } Z = d_1^- + d_1^+$$

*Subject to the constraint*

$$80x_1 + 40x_2 + d_1^- - d_1^+ = 800,$$

$$x_1 + x_2 \le 10$$

$$\text{and } x_1, \quad x_2, \quad d_1^-, \quad d_1^+ \ge 0.$$

**5.** Formulate the problem given in above question-4, as a GP problem with the following equally ranked goals

(i) to earn a profit of Rs. 800 per week

(ii) beacause of the limited sales capacity the maximum number of chairs and lamps that can be sold are 6 and 8 per week respectively.

**Answer:** *The GP can be stated as:*

$$\text{Minimize } Z = d_1^- + d_2^- + d_3^-$$

*Subject to the constraint*

$$80x_1 + 40x_2 + d_1^- - d_1^+ = 800,$$

$$x_1 + d_2^- = 6$$

$$x_1 + d_3^- = 8$$

$$\text{and } x_1, \quad x_2, \quad d_1^-, \quad d_1^+, \quad d_2^-, \quad d_3^- \ge 0.$$

**6.** Suppose a firm manufactures two products. Each product requires time in two production departments: Product 1 requires 20 hours in department 1 and 10 hours in department 2. Product 2 requires 10 hours in department 1 and 10 hours in department 2. Production time is limited in department 1 to 60 hours and in department 2 to 40 hours. Contribution to profits for the two products in Rs. 40 and Rs. 80 respectively. Management has established the following goal priorities:

$P_1$ (priority1): To meet production goals of 2 units for each product.

$P_2$ (priority2): To maximize profits.

Formulate the problem.

**Answer:** $\text{Minimize } Z = P_1 d_2^- + P_1 d_3^- + P_2 d_1^-$

*Subject to the constraint*

$20x_1 + 80x_2 + d_1^- - d_1^+ = 1000,$

$x_1 + d_2^- - d_2^+ = 2$

$x_1 + d_3^- - d_3^+ = 2$

*and* $x_1, \quad x_2, \quad d_1^-, \quad d_1^+, \quad d_2^-, \quad d_2^+, \quad d_3^-, \quad d_3^+ \geq 0.$

**7.** A company manufactures two products radios and transistors which must be processed through assembly and finishing departments. Assembly has 90 hours available, finishing can handle upto 72 hours of work. Manufacturing one radio requires 6 hours in assembly and 3 hours in finishing. Each transistor requires 3 hours in assembly and 6 hours in finishing. If profit is Rs. 120 per radio and Rs. 90 per transistor, determine the best combination of radio and transistors to realize a maximum profit of Rs. 2000.

Formulate the problem as a GP problem. Also solve the GP problem by graphical as well as by modified simplex method.

**Answer:** *Minimize* $Z = d_1^-$

*Subject to the constraints*

$120x_1 + 90x_2 + d_1^- - d_1^+ = 2000,$

$6x_1 + 3x_2 \leq 90$

$3x_1 + 6x_2 \leq 72$

*and* $x_1, \quad x_2, \quad d_1^-, \quad d_1^+ \geq 0.$

*where* $x_1$ and $x_2$ *are the number of radios and transistors produced respectively.*

*Optimum solution: No. of radios =* $x_1 = 12$ *and No. of transistors =* $x_2 = 6$.

*Minimize* $Z = Rs.20$

*The profit target is missed by Rs. 20.*
*i.e., The profit earned is Rs. 2000-20 = Rs. 1980*



**8.** In the problem given in question-6, the company sets the following two equally ranked goals
(i) reach a profit goal of Rs. 1500
(ii) meet a product of radios goal of 10

Formulate the problem as a GP problem and solve by graphical as well as by modified simplex method.

**Answer:** *Minimize* $Z = d_1^- + d_2^-$

*Subject to the constraints*

$$120x_1 + 90x_2 + d_1^- - d_1^+ = 1500$$
$$6x_1 + 3x_2 \le 90$$
$$3x_1 + 6x_2 \le 72$$
$$x_1 + d_2^- - d_2^+ = 10$$

*and* $x_1,\ x_2,\ d_1^-,\ d_1^+,\ d_2^-,\ d_2^+ \ge 0.$

*where* $x_1$ and $x_2$ *are the number of radios and transistors produced respectively.*

*Optimum solution: No. of radios =* $x_1 = 25/2$ *and No. of transistors =* $x_2 = 0$.

*Minimize* $Z = 0$

*Thus, profit is exactly Rs. 1500*
*and the goal of 10 radios is better by (25/2)-10=5/2.*

**9.** Find $x_1, x_2,$ to *Minimize* $Z = (d_1^-, d_2^-)$

Subject to the constraints

$$x_1 + x_2 + d_1^- - d_1^+ = 20,$$
$$4x_1 + 5x_2 + d_2^- - d_2^+ = 150$$

and $x_1,\ x_2,\ d_1^-,\ d_1^+,\ d_2^-,\ d_2^+ \ge 0.$

**Answer:** $x_1 = 75/2,\ x_2 = 0,$ *Minimize* $Z = (0,0),\ d_1^+ = 35/2,$
*or* $x_1 = 0,\ x_2 = 30,$ *Minimize* $Z = (0,0),\ d_1^+ = 10.$

**10.** A manufacturing firm produces two types of products A and B. According to past experience, production of either Product A or Product B requires an average of one hour in the plant. The plant has a normal production capacity of 400 hours a month. The marketing department of the firm reports that because of limited market, the maximum number of Product A and Product B that can be sold in a month are 240 and 300 respectively. The net profit from the sale of Product A and Product B are Rs.800 and lbs. 400 respectively. The manager of the firm has set the following goals arranged in the order of importance (pre-emptive priority factors).

$P_1$: He wants to avoid any underutilization of normal production capacity.

$P_2$: He wants to sell maximum possible units of Product A and Product B.

Since the net profit from the sale of Product A is twice the amount from that of Product B, the manager has twice, as much desire to achieve sales for Product A as for Product B.

P3: He wants to minimize the overtime operation of the plant as much as possible.

Solve this problem by Graphical Method of GP as well as by modified simplex method.

***Answer:*** *Minimize* $Z = P_1 d_1^- + P_2(2d_2^- + d_3^-) + P_3 d_1^+$

*Subject to the constraints*

$x_1 + x_2 + d_1^- - d_1^+ = 400$

$6x_1 + d_2^- = 240$

$x_2 + d_3^- = 300$

*and* $x_1, \quad x_2, \quad d_1^-, \quad d_1^+, \quad d_2^-, \quad d_3^- \geq 0.$

*where* $x_1$ and $x_2$ *are the number of units of product A and product B produced.*

*Optimum solution: No. of product A =* $x_1 = 240$ *and No. of product B =* $x_2 = 300$.

*The overtime operation goal could not be achieved. Overtime operation of the plant is overachieved by 140 hours over the normal capacity of 400 hours a month.*

**11.** Find $x_1, x_2,$ to *Minimize* $Z = (3d_1^+ + 2d_2^+, d_3^-, d_4^-)$

Subject to the constraints

$x_1 + x_2 + d_1^- - d_1^+ = 8,$

$x_1 + d_2^- - d_2^+ = 3$

$3x_1 + 5x_2 + d_3^- - d_3^+ = 65$

$x_1 + x_2 + d_4^- - d_4^+ = 65$

and $x_1, \quad x_2, \quad d_1^-, \quad d_1^+, \quad d_2^-, \quad d_2^+, \quad d_3^-, \quad d_3^+, \quad d_4^-, \quad d_4^+ \geq 0.$

***Answer:*** $x_1 = 0, \quad x_2 = 8,$ *Minimize* $Z = (0, 25, 57).$

## 2.11 OBJECTIVE QUESTIONS

### A. Fill in the Blanks.

*Fill in the blanks"……" so that the following statements are complete and correct.*

1. In GP, the deviational variable must satisfy $d_i^- \times d_i^+ = \ldots\ldots$.
2. GP begins with the most important goal and continues until theachievement of a ..........important goal.
3. In GP problem, a constraint having unachieved variable is expressed as a……….than or equal to type constraint.
4. In the simplex method of GP the variable to enter the solution- mix is selected with ..........priority row and most.......... $c_j - Z_j$ value in it.
5. Consider a goal with constraint $g(x_1, x_2, \ldots\ldots, x_n) + d_1^- \geq b_1 (d_1^- \geq 0)$ with $d_1^-$ in the objective function. Then the constraint is active provided………

### B Multiple Choice Questions.

*Indicate the correct answer for each question by writing the corresponding letter from* (a), (b), (c) *and* (d).

6. In GP problem, goals are assigned priorities such that

(a) goals may not have equal priority

(b) higher priority goals must be achieved before lower priority goals.

(c) goals of greatest importance are given lowest priority

(d) None of these.

7. Deviational variables in GP model must satisfy following conditions

(a) $d_i^+ - d_i^- = 0$ (b) $d_i^+ \times d_i^- = 0$

(c) $d_i^+ + d_i^- = 0$ (d) none of these.

8. For applying a GP approach, decision-maker must

(a) set target for each of the goals

(b) assign pre-emptive priority to each goal

(c) assume that linearity exists in the use of resources to achieve goals

(d) all of these.

**9.** In GP at optimality, which of the following conditions indicate that a goal has been exactly satisfied

(a) positive deviational variable is in the solution mix with a negative value

(b) both positive and negative deviational variables are in the solution mix

(c) both positive and negative deviational variables are not in the solution mix

(d) none of these.

**10.** Consider a goal with constraint $g(x_1, x_2, ......, x_n) + d_1^- - d_1^+ = b_1$ and the term $3d_1^- + 2d_1^+$ in the objective function, the decision-maker

(a) prefers $g(x_1, x_2, ......, x_n) \le b_1$ rather than $\ge b_1$

(b) prefers $g(x_1, x_2, ......, x_n) \ge b_1$ rather than $\le b_1$

(c) not concerned with either $\le$ or $\ge 1$

(d) none of these.

### Answers To Objective Questions

**1.** 0 **2.** *less.* **3.** *greater.* **4.** *highest*; *negative.* **5.** $d_i^- > 0$.

**6.** (*b*). **7.** (*b*). **8.** (*d*). **9.** (*c*). **10.** (*b*).

## 2.12 GLOSSARY

- **LP:** Linear Programming is a mathematical modelling technique in which a linear function is maximized or minimized when subjected to various constraints.
- **GP:** Goal Programming isan extension of Linear Programming in which targets are specified for a set of constraints.
- **PEGP:** Premptive Goal Programming,where there is hierarchy of priority levels for the goals, so goal of primary importance receive first priority attention, those of secondary importance receive second priority attention, and so forth (if there are more than two priority levels).
- **NPEGP/WGP:** Non- Premptive Goal Programming/Weighted Goal Programming, where all the goals are roughly comparable importance.

## 2.12 REFERENCES & SUGGESTED BOOKS

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**LESSON 3**

**WAITING LINE MODELS**

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## STRUCTURE

3.1 Learning Objectives
3.2 Introduction
3.3 Basic elements of queuing models
3.4 Role of poisson and exponential distributions
3.5 Symbols and notations used
3.6 Distribution of arrivals
  3.6.1 Arrival Distribution Theorm
3.7 Distribution of interarrival time
3.8 Markovian process of interarrival time
3.9 States of queuing system
  3.9.1 Transient state
  3.9.2 Steady state
  3.9.3 Explosive state
3.10 Some important definitions
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  3.12.1 Model I (M/M/I) : (∞/FCFS)
  3.12.2 Model II (M/M/c) : (∞/FCFS)
  3.12.3 Model III (M/M/I) : (N/FCFS)
  3.12.4 Model IV (M/M/c) : (N/FCFS)
3.13 Applications of queuing theory
3.14 Limitations of queuing theory
3.15 Summary
3.16 Glossary
3.17 Answers to In-text Questions
3.18 Self-Assessment Questions

3.19 Suggested Readings

## 3.1 LEARNING OBJECTIVES

After reading the unit, students will be able to

- Define basic elements of queuing system.
- Describe role of poisson and exponential distributions.
- Explain markovian process of interarrival time.
- Understand the concept of different queuing models.
- Solve the problems related to waiting line models.
- Identify the situation where the waiting line models can be used.

## 3.2 INTRODUCTION

Queues or waiting lines are a typical occurrence in both regular life and several corporate and industrial settings. When the ability to deliver that service is now unable to meet the current demand, it happens. Additionally, because clients arrive in a random pattern, queues arise when the service rate is larger than the arrival rate. The most frequent locations in daily life where lines form are: movie ticket booths, bank counters, railroad reservation counters, phone booths, doctor's offices, gas stations, post offices, etc.

In addition to these, queues also form in the manufacturing sector in instances where goods are waiting for the next step in the process or waiting to be moved to another location, machines are waiting for repair parts to be assembled in assembly lines, workers are waiting at the tool crib to obtain tools, etc. This could lengthen the production cycle, raising the product's cost, and it might push back the delivery date. Because the findings are frequently employed when making business decisions regarding the resources required to deliver service, queuing theory is regarded as one of the standard approaches of operations research and management science (along with linear programming, simulation, etc.).

The mathematical analysis of queues is known as queueing theory. The theory makes it possible to mathematically analyse a number of connected processes, such as getting to the front of the line, waiting in line, and receiving service. In order to reduce the average cost of using the queuing system and the cost of service, the queuing model aims to determine the ideal service rate and server count. Numerous further mathematical models for understanding and resolving issues with waiting lines are provided by queuing theory.

## 3.3 ELEMENTS OF COST

The units requiring service enter the queuing system on their arrival and join a queue. The queue represents the number of customers waiting for service. A queue is called finite if the number of units in it is finite otherwise it is called infinite. Some of the basic elements of queuing system are as follows:

- Input source of queue
- Queue discipline (Service discipline)
- Service mechanism (Service system)
- System output

### 3.3.1 Input source of queue:

Customers requiring service are generated at different times by an input source, commonly known as population. The rate at which customers arrive at the service facility is determined by the arrival process. An input source is characterized by:

- Size of the calling population
- Pattern of arrivals at the system
- Behavior of the arrivals (Customer behavior)

**a) Size of the calling population**

The size represents the total number of potential customers who will require service. The size of the population is described by the following factors given below:

**i)** **According to source-** The source of customers can be finite or infinite. For example, all people of a city or state (and others) could be the potential customers at a supermarket. The number of people being very large, it can be taken to be infinite whereas there are many situations in business and industrial conditions where we cannot consider the population to be infinite; it is finite.

ii) **According to numbers-** The customers may arrive for service individually or in groups. Single arrivals are illustrated by patients visiting a doctor, students reaching at a library counter etc. On the other hand, families visiting restaurants, ships discharging cargo at a dock are examples of group or batch arrivals.

iii) **According to time-** Customers arrive in the system at a service facility according to some known schedule or else they arrive randomly. Arrivals are considered random when they are independent of one another and their occurrence cannot be predicted exactly. The queuing models wherein customer's arrival times are known with certainty are categorized as deterministic models and are easier to handle. On the other hand, a substantial majority of the queuing models are based on the premise that the customers enter the system stochastically, at random points in time.

**b) Pattern of arrivals at the system**

Customers' arrival processes (or patterns) at the support system are divided into two groups: static arrival processes and dynamic arrival processes.

i) **In static arrival process**, the control depends on the nature of arrival rate (random or constant). Random arrivals are either at a constant rate or varying with time. Thus to analyze the queuing system, it is necessary to attempt to describe the probability distribution of arrivals. From such distributions we obtain average time between successive arrivals, also called "*inter-arrival time*" (time between two consecutive arrivals), and the average arrival rate (i.e. number of customers arriving per unit of time at the service system).

ii) **In dynamic arrival process**both the service centre and the customers have control. By varying staffing levels at various service times, varying service fees at various times, or allowing entrance with appointments, the service facility can adjust its capacity to match changes in the intensity of demand.

Frequently in queuing problems, the number of arrivals per unit of time can be estimated by a probability distribution known as the Poisson distribution, as it adequately supports many real world situations.

**c) Behavior of arrivals (Customer behavior)**

The behaviour or attitude of the customers entering the queueing system is another factor to take into account. Customers can be divided into two groups based on how patient or impatient they are. A customer is described as patient if, upon entering the service system, he remains there until served, regardless of how long he must wait. In contrast, an impatient customer is one who waits in the queue for a predetermined amount of time before leaving due to factors like the length of the queue in front of him.Some interesting observations of customer behavior in queues are as follows:

i) **Balking-** Some customers even before joining the queue get discouraged by seeing the number of customers already in service system or estimating the excessive waiting time for desired service decide to return for service at a later time. This is known as balking.

ii) **Reneging-** Customers after joining the queue wait for sometime and leave the service system due to intolerable delay, so they renege.

iii) **Jockeying-** Customers who switch from one queue to another hoping to receive service more quickly are said to be jockeying.

iv) **Collusion-** Customers in the queue may demand the service on their behalf as well as on behalf of others is known as collusion.

### 3.3.2 Queue discipline (Service discipline):

In the queue structure, the important thing to know is the queue discipline. The queue discipline is the order or manner in which customers from the queue are selected for service. There are a number of queue disciplines in which customers in the queue are served. Some of these are as follows:

**(a) Static queue disciplines** are based on the individual customer's status in the queue. Few of such disciplines are:

i) **First-come-first-served (FCFS)-** If the customers are served in the order of their arrival, then this is known as the first-come-first-served (FCFS) service discipline.

ii) **Last-come-first-served (LCFS)-** Sometimes, the customers are serviced in the reverse order of their entry so that the ones who join the last are served first, then this is called last-come-first-served (LCFS) service discipline.

**(b) Dynamic queue disciplines** are based on the individual customer attributes in the queue. Few of such disciplines are:

i) **Service in Random Order (SIRO)-** Under this rule customers are selected for service at random, irrespective of their arrivals in the service system. In this every customer in the queue is equally likely to be selected. The time of arrival of the customers is, therefore, of no relevance in such a case.

iii) **Priority Service-** Under this rule customers are grouped in priority classes on the basis of some attributes such as service time or urgency or according to some identifiable characteristic to provide the service. The treatment of VIPs in preference to other patients in a hospital is an example of priority service.

iv) **Round Robin service-** Every customer gets a time slice. If his service is not completed, he will re-enter the queue.

Use simple and easily understandable language.

### 3.3.3 Service mechanism (Service system):

The uncertainties involved in the service mechanism are the number of servers, the number of customers getting served at any time, and the duration and mode of service. Networks of queues consist of more than one servers arranged in series and/or parallel. Random variables are used to represent service times, and the number of servers, when appropriate. If service is provided for customers in groups, their size can also be a random variable. A service system has only a few components listed below:

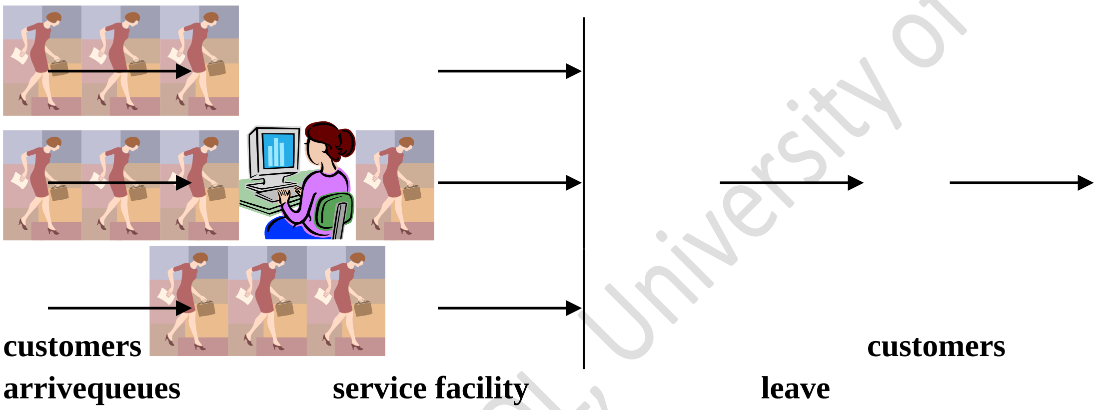
- Configuration of the service system
- Speed of the service
- System capacity

**a) Configuration of the service system**

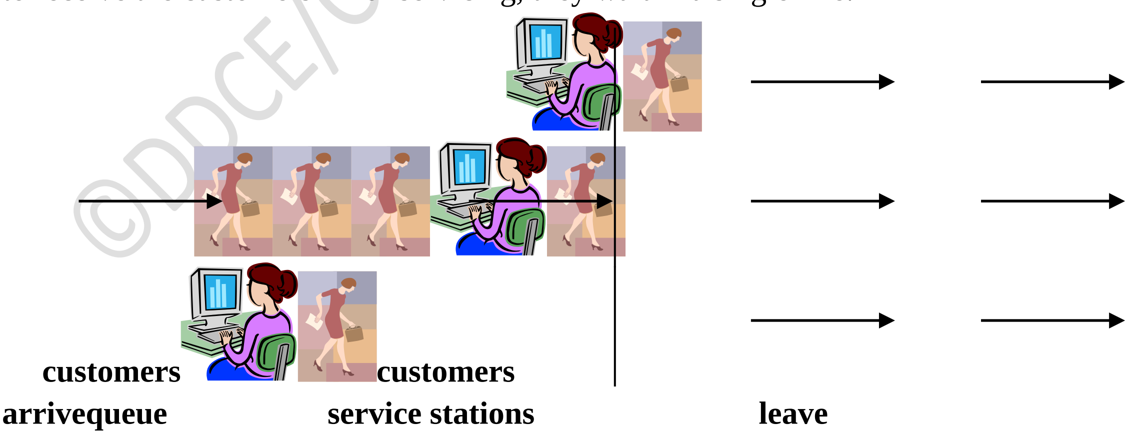
The conditions of the queue determine how customers enter the support system. If the server is not busy when the clients arrive, they are served right away. If not, the client is prompted to join the queue, which can be set up in a variety of ways. The configuration of the service system refers to the physical layout of the service centres. Typically, service systems are categorised according to their amount of servers or channels:

i) **Single Server – Single Queue-**Single server models are those where there is only one queue and one service station facility, and the customer waits until the service point is prepared to accept him in for servicing. A library counter serving as an example of a single server facility, with students gathering at it.

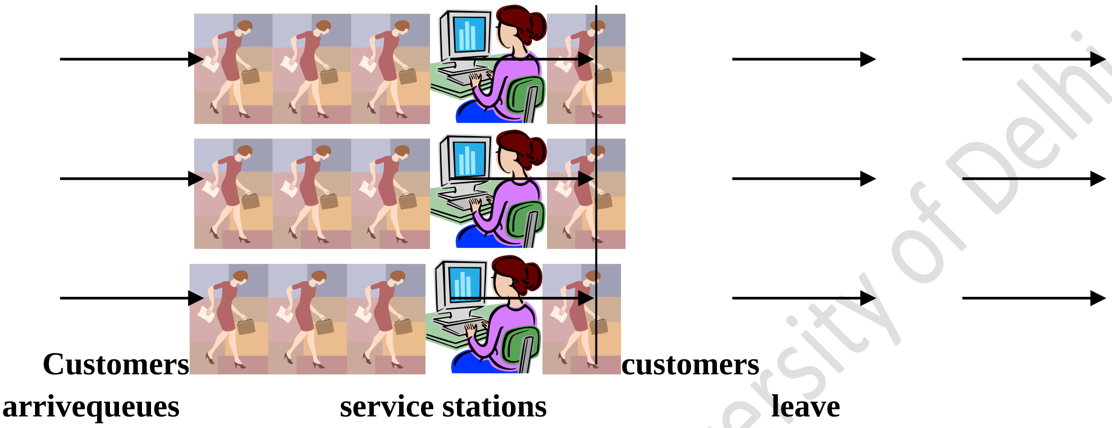

**ii) Single Server – Several Queues-** In this type of facility there are several queues and the customer may join any one of these but there is only one service channel.




**iii) Several (Parallel) Servers – Single Queue-**This kind of strategy uses multiple servers, each of which offers the same kind of service. When one of the service channels is prepared to receive the customers in for servicing, they wait in a single line.



**iv) Several Servers – Several Queues-**This kind of model comprises of a number of servers, each of which has its own queue. An illustration of this type of model is the various cash counters in an electricity office where customers can settle their electricity bills.




**v) Service facilities in a series-**In this, a customer approaches the first station, receives some service there, moves to the next station, receives more service there, and then does it all over again and so forth, until the user ultimately exits the system after receiving the full service. For instance, the machining of a particular steel object might involve a succession of single servers performing cutting, turning, knurling, drilling, grinding, and packaging operations.


**b) Speed of Service**

In a queuing system, the speed with which service is provided can be expressed in either of two ways as, service rate and as service time.

- **i)** **The service rate** describes the average number of customers that can be served per unit time. Service rate is denoted by $\mu$.
- **ii)** **The service time** indicates the amount of time needed to serve a customer. Service time is the reciprocal of service rate, i.e, service time = $1/\mu$.

**Eg:** If a cashier can attend, on an average 10 customers in an hour, the service rate would be expressed as 10 customers/hour and service time would be equal to 6 minutes/customer.

**c) System capacity**

In a queuing system, it is important to take into account how many consumers can wait at once. If the waiting area is big, it can be assumed that it is practically infinite. But based on our regular use of telephone networks, we know that the size of the buffer that receives our call while we wait for a free line is crucial as well.

### 3.3.4 System output:

The rate at which consumers are served is referred to as system output. It depends on how long the facility needs to provide the service and how the service facility is set up. In a single channel facility, the queue's output is unimportant since the client leaves after obtaining the service. However, in a multistage channel facility, the queue's output is crucial because the probability of a service station breakdown can affect the queues. The queue prior to the breakdown will get longer, while the line after the breakdown will get shorter.

## 3.4 ROLE OF POISSON AND EXPONENTIAL DISTRIBUTIONS

Constructive queuing models are analytically reliable representations of real-world systems. These two requirements are frequently satisfied by a queuing model based on the Poisson process and its companion exponential probability distribution. A Poisson process simulates the emergence of random events from a memoryless process, such as the arrival of a customer, a web server's request for action, or the accomplishment of the requested actions. That is, the amount of time that will pass from the present moment till the next event does not depend on when the previous event occurred. The observer counts the number of events that take place within a period of defined length while calculating the Poisson probability distribution.The observer keeps track of how much time passes between consecutive events in the (negative) exponential probability distribution. The underlying physical process is memoryless in both cases.

Inputs from the environment are frequently handled by models based on the Poisson process in a way that closely resembles how the system being represented would handle the same inputs. The resulting analytically obedient models provide insight into the system being modelled as well as the shape of their solution. Even a Poisson-based queuing model that performs comparably poorly in simulating the specific system performance can be helpful. System designers that like to incorporate a security aspect in their designs are attracted by the fact that such models frequently give evaluations of "worst-case" situational scenarios.Additionally, the shape of a queuing problem's solution, whose precise behaviour is poorly replicated, is frequently revealed by studying models based on the Poisson process. As a result, queuing models are commonly represented by the exponential distribution as Poisson processes.

The exponential distribution with parameter $\lambda$ is given by $\lambda e^{-\lambda t}$ for $t \geq 0$. If T is a random variable that represents interarrival times with the exponential distribution then,

$$P(T \leq t) = 1 - e^{-\lambda t}$$

and $$P(T > t) = e^{-\lambda t}.$$

This distribution lends itself well to modeling customer interarrival times or service times for a number of reasons. The first is the fact that the exponential function is a strictly decreasing function of t. This means that after an arrival has occurred, the amount of waiting time until the next arrival is more likely to be small than large. Another significant property of the exponential distribution is what is known as the no-memory property. The no-memory property suggests that the time until the next arrival will never depend on how much time has already passed. This makes intuitive sense for a model where we're measuring customer arrivals because the customers' actions are clearly independent of one another.

The Poisson distribution is used to calculate the likelihood that a specific number of entries will occur within a specific time frame. The Poisson distribution with parameter $\lambda$ is given by,

$$\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Where n is the number of arrivals.

We find that if we set n = 0, the Poisson distribution gives us $e^{-\lambda t}$, which is equal to $P(T > t)$ from the exponential distribution.

**IN-TEXT QUESTIONS**

1. The objective of the queuing model is to find out the ___________ service rate and the number of servers so that the average cost of being in the queuing system and the cost of service are minimized.
2. Which of the following is/are the basic elements of queuing system?
   (a) Queue discipline
   (b) Service mechanism
   (c) System output
   (d) All of above
3. Some customers decide to return for assistance at a later time after becoming discouraged by the number of people already in the queue or the estimated lengthy wait time before even getting in line. This is known as:
   (a) Jockeying
   (b) Balking
   (c) Reneging
   (d) Collusion
4. The _____________ is used to determine the probability of a certain number of arrivals occurring in a given time period.



## 3.5 SYMBOLS AND NOTATIONS USED

The following are some symbols and terminology used in the queuing models:

| | | |
|---|---|---|
| $n$ | = | Number of units in the system |
| $P_n(t)$ | = | Transient state probability of n units in the system at time t |
| $E_n$ | = | State in which n units in the system |
| $P_n$ | = | Steady state probability of having n units in the system |
| $\lambda_n$ | = | Mean arrival rate |

| | | |
|---|---|---|
| $\mu_n$ | = | Mean service rate |
| $\lambda$ | = | Mean arrival rate when $\lambda_n$ is constant for all n |
| $\mu$ | = | Mean service rate when $\mu_n$ is constant for all $n \geq 1$ |
| c | = | Number of parallel service stations |
| $\rho$ | = | Traffic intensity $(\lambda/\mu)$ |
| $\Phi_T(n)$ | = | Probability of n services in time T |
| $\psi(w)$ | = | Probability density function of waiting time in the system |
| $L_s$ | = | Expected no. of units in the system (Length of the system) |
| $L_q$ | = | Expected no. of units in the queue (Length of the queue) |
| $W_s$ | = | Expected waiting time per customer in the system |
| $W_q$ | = | Expected waiting time per customer in the queue |
| $(W/W > 0)$ | = | Expected waiting time of a customer who has to wait |
| $(L/L > 0)$ | = | Expected length of non-empty queues |
| $P(W > 0)$ | = | Probability of a customer having to wait for service |

## 3.6 DISTRIBUTION OF ARRIVALS

### 3.6.1 Arrival Distribution Theorm:

If the arrivals are completely random then the probability distribution of number of arrivals in a fixed time interval follows a poisson's distribution.

**Proof:** To prove this theorem we shall make some assumptions which are as follows: Let there are n units in the system at time t.

**1)** The probability of one arrival in small time interval $\Delta t = \lambda.\Delta t$

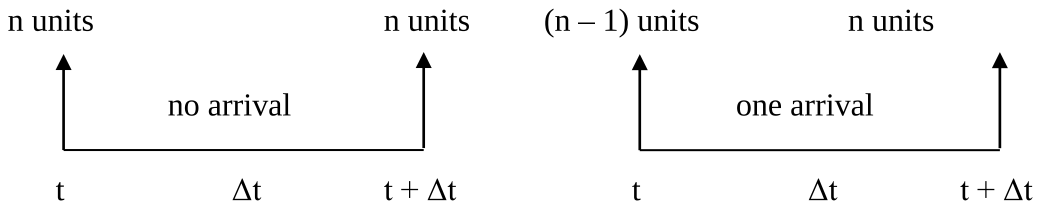
**2)** The probability of more than one arrival in the time interval $\Delta t$ is zero because $\Delta t$ is very small.

**3)** The process has independent increments.

4) $P_n(t)$ be the probability of n arrivals in time t.

There may be two cases:

**Case-I:** when $n > 0$ then, two events can occur, which are shown below:



Let there are n units in the system at time t, no arrival takes place in time $\Delta t$. Hence there remain n units in the system at time $(t + \Delta t)$.

Probability of this event = (probability of n units in the system) x (probability of no arrival in time $\Delta t$)

$$= P_n(t) \,.\, (1 - \lambda.\Delta t)$$

Let there are $(n - 1)$ units in the system at time t, one arrival takes place in time $\Delta t$. So there are n units in the system at time $(t + \Delta t)$.

Probability of this event = [probability of $(n - 1)$ units in the system] x (probability of one arrival in time $\Delta t$)

$$= P_{n-1}(t) \,.\lambda.\Delta t$$

Hence the probability of n units in the system at time $(t + \Delta t)$ is,

$$P_n(t + \Delta t) = P_n(t) \,.\, (1 - \lambda.\Delta t) + P_{n-1}(t) \,.\lambda.\Delta t$$

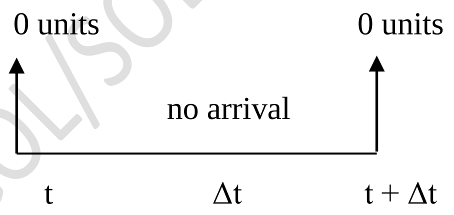
$$P_n(t + \Delta t) = P_n(t) - \lambda.\, P_n(t).\, \Delta t + P_{n-1}(t) \,.\lambda.\Delta t$$

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -\lambda.\, P_n(t) + \lambda.\, P_{n-1}(t)$$

Taking limit $\Delta t \to 0$, we get

$$P_n'(t) = -\lambda.\, P_n(t) + \lambda.\, P_{n-1}(t) \quad , \text{for } n > 0 \quad \ldots\ldots\ldots(1)$$

**Case II:** When n = 0 then,




Let there is no unit in the system at time t, no arrival takes place in time $\Delta t$. Hence there would be zero units in the system at time $(t + \Delta t)$.

Probability of this event = (probability of no unit in the system) x (probability of no arrival in time $\Delta t$)

$$P_0(t + \Delta t) = P_0(t) \,.\, (1 - \lambda.\Delta t)$$

$$P_0(t + \Delta t) - P_0(t) = -\lambda.\, P_0(t).\, \Delta t$$

$$\frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda.\, P_0(t)$$

Taking limit $\Delta t \to 0$, we get

$$P_0'(t) = -\lambda.\, P_0(t) \quad , \text{for } n = 0 \quad \ldots\ldots\ldots(2)$$

In order to find the probability distribution, we shall make use of generating function of $P_n(t)$. i.e,

$$P(z,t) = \sum_{n=0}^{\infty} P_n(t).\, z^n \qquad \ldots\ldots\ldots(3)$$

Differentiating with respect to t, we get

$$P'(z,t) = \sum_{n=0}^{\infty} P_n'(t).\, z^n \qquad \ldots\ldots\ldots(4)$$

Multiplying equation (1) by $z^n$ and taking summation from n = 1 to n = ∞, we get

$$\sum_{n=1}^{\infty} z^n P_n'(t) = -\lambda \sum_{n=1}^{\infty} z^n P_n(t) + \lambda \sum_{n=1}^{\infty} z^n P_{n-1}(t)$$

Adding equation (2) to the above equation, we get

$$\sum_{n=1}^{\infty} z^n P_n'(t) + z^0 P_0'(t) = -\lambda \sum_{n=1}^{\infty} z^n P_n(t) - \lambda z^0 P_0(t) + \lambda \sum_{n=1}^{\infty} z^n P_{n-1}(t)$$

or,

$$\sum_{n=0}^{\infty} z^n P_n'(t) = -\lambda \sum_{n=0}^{\infty} z^n P_n(t) + \lambda \sum_{n=1}^{\infty} z^n P_{n-1}(t)$$

Making use of equation (3) and (4), we get

$$P'(z,t) = -\lambda P(z,t) + \lambda P(z,t).\, z$$

or, $$P'(z,t) = \lambda P(z,t)\,(z-1)$$

or, $$\frac{P'(z,t)}{P(z,t)} = \lambda\,(z-1)$$

Integrating with respect to t, we get

$$\log P(z,t) = \lambda\,(z-1).\, t + c$$

Putting t = 0,

$$\log P(z,0) = c \qquad \ldots\ldots\ldots(5)$$

Now from equation (3), we have

$$P(z,0) = \sum_{n=0}^{\infty} P_n(0).\, z^n$$

$$P(z,0) = \sum^{\infty} P_n(0).\, z^n + P_0(0).\, z^0$$

n = 1

Since, $P_0(0)$ = Probability of zero unit at time zero= universal truth = 1
and $P_n(0)$ = Probability of n unit at time zero = 0

Therefore, $P(z,0) = 1 + 0 = 1$

Now from equation (5),

$$c = \log(1) = 0$$

Hence, $\log P(z,t) = \lambda(z-1).t$

$P(z,t) = e^{\lambda(z-1).t}$

or, $P(z,t) = e^{\lambda z t}. e^{-\lambda t}$ ……….(6)

Now from equation (3), we have

$$P(z,t) = \sum_{n=0}^{\infty} P_n(t). z^n$$

Where, $P_n(t) = \frac{1}{n!}\left(\frac{d^n P(z,t)}{dz^n}\right)_{z=0}$

Therefore using equation (6), we have

$$P_n(t) = \frac{1}{n!}\left(\frac{d^n e^{\lambda z t}. e^{-\lambda t}}{dz^n}\right)_{z=0}$$

$$= \frac{e^{-\lambda t}}{n!}\left(\frac{d^n e^{\lambda z t}}{dz^n}\right)_{z=0}$$

$$= \frac{e^{-\lambda t}}{n!}\left(e^{\lambda z t}(\lambda t)^n\right)_{z=0}$$

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Which is poisson’s distribution, hence the theorem.

## 3.7 DISTRIBUTION OF INTERARRIVAL TIME

**Theorem:** If n, the number of arrivals in time t, follow the Poisson’s distribution,

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

then T, interarrival time obeys the negative exponential law,

$$a(T) = \lambda e^{-\lambda t}$$

**Proof:** Given that the arrivals follows the Poisson’s distribution, i.e,

$$P_n(t) = \frac{e^{-\lambda t}\,(\lambda t)^n}{n!} \qquad \ldots\ldots\ldots(1)$$

If $(T + \Delta T)$ be the interarrival time then putting $t = T + \Delta T$ and $n = 0$, we get

$$P_0(T + \Delta T) = e^{-\lambda(T + \Delta T)}$$
$$= e^{-\lambda T}\, e^{-\lambda.\,\Delta T}$$

Expanding $e^{-\lambda.\,\Delta T}$ up to first approximation because $\Delta T$ is very small, we get

$$P_0(T + \Delta T) = e^{-\lambda T}\,[1 - \lambda.\,\Delta T] \qquad \ldots\ldots\ldots(2)$$

Again if T be the inter arrival time then putting $t = T$ and $n = 0$ in equation (1), we get

$$P_0(T) = e^{-\lambda T} \qquad \ldots\ldots\ldots(3)$$

Using this equation (2) implies,

$$P_0(T + \Delta T) = P_0(T)\,[1 - \lambda.\,\Delta T]$$

or,

$$P_0(T + \Delta T) = P_0(T) - P_0(T).\lambda.\,\Delta T$$
$$\frac{P_0(T + \Delta T) - P_0(T)}{\Delta T} = -\lambda.P_0(T) = -\lambda.\,e^{-\lambda T}$$

Taking limit $\Delta t \rightarrow 0$, we get

$$P_0{}'(T) = -\lambda.\,e^{-\lambda T}$$

Hence by the definition of probability density function, we have

$$a(T) = \lambda e^{-\lambda T}$$

Omitting the negative sign because probability density function is always positive.
Hence the theorem.

## 3.8 MARKOVIAN PROCESS OF INTERARRIVAL TIME

**Theorem:** According to the Markovian process of interarrival time, the amount of time until the subsequent arrival is made is irrespective of the amount of time that has passed since the previous arrival. i.e, to say,

$$P[T \geq t_1 \mid T \geq t_0] = P[0 \leq T \leq t_1 - t_0]$$

Proof: Using conditional probability, we can write

$$P[T \geq t_1 \mid T \geq t_0] = P[(T \geq t_1) \text{and } (T \geq t_0)]/P[T \geq t_0] \qquad \ldots\ldots\ldots\ldots(1)$$

Since the interarrival times are exponentially distributed, R.H.S. of equation (1) becomes,

$$[\textstyle\int_{t0}^{t1} \lambda.e^{-\lambda t}\,dt]/\,[\int_{t0}^{\infty} \lambda.e^{-\lambda t}\,dt] = [e^{-\lambda t1} - e^{-\lambda t0}]/[- e^{-\lambda t0}]$$

Hence,

$$P[T \geq t_1 \mid T \geq t_0] = 1 - e^{-\lambda(t1 - t0)} \qquad \ldots\ldots\ldots\ldots(2)$$

Also,

$$P[0 \le T \le t_1 – t_0] = \int_0^{(t1-t0)} \lambda.e^{-\lambda t}\, dt = 1 - e^{-\lambda(t1-t0)} \ldots\ldots\ldots\ldots(3)$$

From equations (2) and (3), we have

$$P[T \ge t_1 \mid T \ge t_0] = P[0 \le T \le t_1 – t_0]$$

This is the markovian process of interarrival time.

## 3.9 STATES OF QUEUING SYSTEM

The analysis of queuing theory involves the study of the behavior of the system over time. The state of the system is the basic concept in the analysis of the queuing theory. The state of the queuing system may be classified as follows:

### 3.9.1 Transient state:

When a system's operational features rely on time, it is said to be in a transitory state. For example, a queuing system is said to be in a transient state when the likelihood of having clients in the system depends on time. This typically happens towards the beginning of the system's operation, when it starts to go through a number of changes. But after some time, it becomes stable.

### 3.9.2 Steady state:

A queuing system is said to be in a steady state when the probability of having a certain amount of customers is independent of time. When a system's working characteristics become timely independent, it is said to be in steady state. This typically occurs as a device ages.

### 3.9.3 Explosive state:

The length of the queue will grow over time and eventually reach infinity if the system's arrival rate is higher than its service rate. Such a state is known as explosive state.

## 3.10 SOME IMPORTANT DEFINITIONS

**(i) System length-** The average number of customers in the system, those waiting to be and those being serviced, is known as the length of the system.

**(ii) Queue length-**Queue length is the average number of customers in line waiting to obtain service.

**(iii) Waiting time in the queue-**It is the typical length of time a customer must wait in line before it is put into operation.

**(iv) Waiting time in the system-** It is the amount of time, on average, that a consumer spends using the system between joining the queue and receiving their service.

**(v) Servicing time-** The time taken for servicing of a unit is known as its servicing time.

**(vi) Mean arrival rate-** The expected number of arrivals occurring in the time interval of length unity is called mean arrival rate. It is denoted by $\lambda$.

**(vii) Mean arrival time-** It is the reciprocal of mean arrival rate and is defined as, Mean arrival time = 1/mean arrival rate = $1/\lambda$.

**(vii) Mean servicing rate-** The expected number of services completed in the time interval of length unity is called mean servicing rate. It is denoted by $\mu$.

**(vii) Mean servicing time-** It is the reciprocal of mean servicing rate and is defined as, Mean servicing time = 1/mean servicing rate = $1/\mu$.

**(ix) Server busy period-** The busy period of the server is the time during which itremains busy in servicing.

**(x) Server idle period-** When all the units in the queue are served, the idle period of the server begins and it continues up to the time of the arrival of the unit i.e, the idle period of the server is the time during which he remains free because there is no unit in the system to be served.

**(xi) Traffic intensity (Utilization factor)-** An important parameter in any queuing system is the traffic intensity also called the load or the utilization factor, defined as the ratio of the mean servicing time over the mean arrival time. It is denoted by $\rho$ and is defined as, $\rho = \lambda/\mu$

## 3.11 KENDALL - LEE NOTATIONS

Notation for describing the characteristics of a queuing model was first suggested by David G. Kendall in 1953. Kendall's notation introduced an (a/b/c) queuing notation that can be found in all standard modern works on queuing theory.

Where,

a describes the interarrival time distribution,
b the service time distribution and
c the number of servers

The symbols conventionally used for a and b are:

M for exponential distribution (M stands for Markov),
D for deterministic distribution and
G (or GI) for general distribution

For example, "G/D/1" would indicate a General arrival process, a Deterministic (constant time) service process and a single server. Some other examples are M/M/1, M/M/c, M/G/1, G/M/1 and M/D/1.

Later in 1966, A. Lee extended Kendall's notations by adding fourth (d) and fifth (e) characteristics to the notation to cover other queuing models. Then the following symbolic expression can be used to fully specify the queuing model:

(a/b/c) : (d/e)

Where a, b and c describes their usual meaning and the addition letters d and e describes the capacity of the system and the queue discipline respectively.

For example, (M/M/4) : (25/FCFS) could represent a bank with exponential arrival times, exponential service times, 4 tellers, total capacity of 25 customers and an FCFS queue discipline.



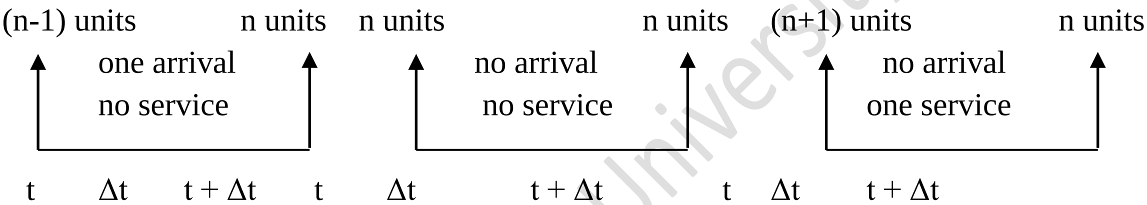
## 3.12 POISSON QUEUES

### 3.12.1 MODEL - I [(M/M/1):(∞/FCFS)] (Birth and Death Model):

Assumptions for this model are as follow:
(i) Both arrival and service rate are independent of the number of customers in the queue.
(ii) The arrival occur completely at random according t the Poisson distribution.
(iii) Only one queue and one service facility is available.
(iv) Capacity of the system is infinite.
(v) Customers are served on the basis of first come first serve service mechanism.
(vi) $\lambda$ = Arrival rate
(vii) $\mu$ = Servive rate
(viii) $\rho = \lambda / \mu$
(ix) $P_n(t)$ = The probability of n units in the system at time t

❖ To obtain the system of steady state equations

There are n > 0 units in the system at time t + $\Delta$t, in the following ways:




Let there are (n-1) units in the system at time t, one arrival takes place in time $\Delta$t and no service provided in time $\Delta$t. Hence there remain n units in the system at time (t + $\Delta$t).

Probability of this event = [probability of (n-1) units in the system]
x (probability of one arrival in time $\Delta$t)
x (probability of no service in time $\Delta$t)
$= P_{n-1}(t) .\lambda.\Delta t . (1 - \mu.\Delta t)$

Let there are n units in the system at time t, no arrival takes place in time $\Delta$t and no service provided in time $\Delta$t. So there are n units in the system at time (t + $\Delta$t).

Probability of this event = [probability of n units in the system]
x (probability of no arrival in time $\Delta$t)
x (probability of no service in time $\Delta$t)
$= P_n(t) . (1- \lambda.\Delta t). (1 - \mu.\Delta t)$

Let there are (n+1) units in the system at time t, no arrival takes place in time $\Delta$t and one service provided in time $\Delta$t.

So there are n units in the system at time (t + $\Delta$t).

Probability of this event = [probability of n units in the system]
x (probability of no arrival in time $\Delta$t)
x (probability of no service in time $\Delta$t)

$= P_{n+1}(t) \,.\, (1- \lambda.\Delta t).\ \ \mu.\Delta t$

All these cases are mutually exclusive, hence the probability of n units in the system at time $(t + \Delta t)$ is,

$$P_n(t + \Delta t) = P_{n-1}(t) \,.\, \lambda.\Delta t \,.\, (1 - \mu.\Delta t) + P_n(t) \,.\, (1- \lambda.\Delta t).\, (1 - \mu.\Delta t) + P_{n+1}(t) \,.\, (1- \lambda.\Delta t).\ \mu.\Delta t$$

Since $\Delta t$ is very small, hence neglecting the terms containing $(\Delta t)^2$

$$P_n(t + \Delta t) = P_{n-1}(t).\lambda.\ \Delta t + P_n(t) - P_n(t).\ \lambda.\Delta t - P_n(t).\ \mu.\Delta t + P_{n+1}(t).\ \mu.\Delta t$$

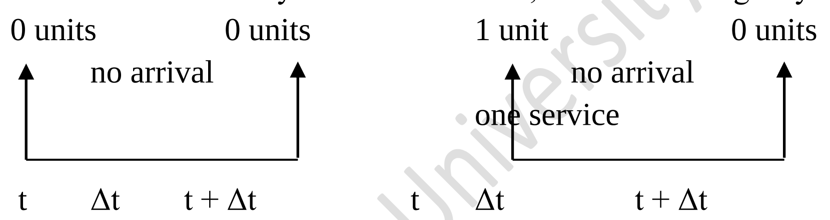
$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \lambda.\ P_{n-1}(t) - \lambda.\ P_n(t) - \mu.\ P_n(t) + \ \mu.\ P_{n+1}(t)$$

$$= \lambda.\ P_{n-1}(t) - (\lambda+\mu).\ P_n(t) + \ \mu.\ P_{n+1}(t)$$

Taking limit $\Delta t \rightarrow 0$, we get

$$P_n'(t) = \lambda.\ P_{n-1}(t) - (\lambda+\mu).\ P_n(t) + \ \mu.\ P_{n+1}(t), \ \text{ for } n > 0 \qquad \ldots\ldots\ldots(1)$$

Similarly, there are n=0 units in the system at time $t + \Delta t$, in the following ways:




Let there is no unit in the system at time t and no arrival takes place in time $\Delta t$.
So there is no units in the system at time $(t + \Delta t)$.

Probability of this event = [probability of 0 unit in the system]
x (probability of no arrival in time $\Delta t$)
$= P_0(t) \,.\, (1- \lambda.\Delta t)$

Let there is one unit in the system at time t, no arrival takes place in time $\Delta t$ and one service provided in time $\Delta t$.
So there is no unit in the system at time $(t + \Delta t)$.
Probability of this event = [probability of one unit in the system]
x (probability of no arrival in time $\Delta t$)
x (probability of one service in time $\Delta t$)
$= P_1(t) \,.\, (1- \lambda.\Delta t).\ \mu.\Delta t$

All these cases are mutually exclusive, hence the probability of no unit in the system at time $(t + \Delta t)$ is,

$$P_0(t + \Delta t) = P_0(t) \,.\, (1- \lambda.\Delta t) + P_1(t) \,.\, (1- \lambda.\Delta t).\ \mu.\Delta t$$

Since $\Delta t$ is very small, hence neglecting the terms containing $(\Delta t)^2$

$$P_0(t + \Delta t) = P_0(t) - P_0(t).\lambda.\Delta t + P_1(t).\ \mu.\Delta t$$

$$\frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = - \lambda. P_0(t) + \mu. P_1(t)$$

Taking limit $\Delta t \rightarrow 0$, we get

$$P_0'(t) = - \lambda. P_0(t) + \mu. P_1(t), \quad \text{for } n = 0 \quad \ldots\ldots\ldots(2)$$

In steady state, the probabilities are independent of time, therefore equations (1) & (2) becomes

$$\lambda. P_{n-1} - (\lambda+\mu). P_n + \mu. P_{n+1} = 0 \quad \ldots\ldots\ldots(3)$$

$$- \lambda. P_0 + \mu. P_1 = 0 \quad \ldots\ldots\ldots(4)$$

These equations are called steady state equations.

❖ Now to solve the system of steady state equations

from equation (4)

$$P_1 = \lambda. P_0 / \mu \quad \ldots\ldots\ldots(5)$$

Since, $\rho = \lambda / \mu$ ……….(6)

Therefore, $P_1 = \rho. P_0$

Put n = 1 in equation (3), we get

$$\lambda. P_0 - (\lambda+\mu). P_1 + \mu. P_2 = 0$$

using (5) & (6), we get

$$P_2 = \rho^2. P_0$$

Similarly, $P_3 = \rho^3. P_0$

$$P_4 = \rho^4. P_0$$

-----------------

$$P_n = \rho^n. P_0 \quad \ldots\ldots\ldots(7)$$

To find $P_0$, use the fact that the total probability is 1, therefore

$$\sum_{n=0}^{\infty} P_n = P_0 + P_1 + P_2 + P_3 + \ldots\ldots\ldots\text{upto } \infty = 1$$

$$P_0 + \rho. P_0 + \rho^2. P_0 + \rho^3. P_0 + \ldots\ldots\ldots\text{upto } \infty = 1$$

$$P_0[1/(1- \rho)] = 1$$

$$P_0 = (1- \rho)$$

From (7), $P_n = (1- \rho).\rho^n$

Hence the steady state distribution of arrivals is, $P_n = (1- \rho).\rho^n$

Also, $P[\text{Queue size} \geq N] = \sum_{n=0}^{\infty} P_n - \sum_{n=0}^{N-1} P_n = 1- (P_0 + P_1 + \ldots\ldots + P_{N-1})$

Using values of $P_0, P_1, \ldots\ldots P_{N-1}$, we get

$$P[\text{Queue size} \geq N] = (\lambda/\mu)^N = \rho^N$$

**Important formulae of Model - I**

- $L_q$= Expected number of units in the queue = $\lambda^2/[\mu.(\mu-\lambda)] = \lambda. W_q$
- $L_s$ = Expected number of units in the system = $\lambda/(\mu-\lambda) = \lambda. W_s$
- $(L/L > 0)$ = Expected length of non-empty queue = $\mu/(\mu-\lambda) = \mu.W_s$
- $W_q$ = Expected waiting time per customer in the queue = $\lambda/[\mu.(\mu-\lambda)] = L_q/\lambda$
- $W_s$ = Expected waiting time per customer in the system = $1/(\mu-\lambda) = L_s/\lambda$
- $(W/W > 0)$ = Expected waiting time of customer who has to wait = $1/(\mu-\lambda)$
- $\rho$ = Busy period = $\lambda / \mu$
- $P_0$ = Probability that exactly zero units in the system = $(1- \rho) = (1- \lambda/\mu)$
- $P_n$ = Probability that exactly 'n' units in the system = $P_0. (\lambda/\mu)^n$
- $P(W > 0)$ = Probability that an arrival will have to wait = $1 - P_0$
- $P(\text{Queue size} \geq n) = (\lambda/\mu)^n$

**Example:** A telephone booth's patrons are thought to be Poisson distributed and come with an average interval of 10 minutes. The phone call's duration is spread exponentially, with a mean of five minutes. Determine:

(a) Expected number of units in the queue.
(b) Expected waiting time in the queue.
(c) Expected number of units in the system.
(d) Expected waiting time in the system
(e) Expected fraction of the day that the phone will be in use.
(f) Probability that the customer will have to wait.

**Solution:** Given,
The mean arrival time = 10 min
The mean service time = 5 min
The mean arrival rate, $\lambda = (1/10) \times 60 = 6/\text{hour}$
The mean service rate, $\mu = (1/5) \times 60 = 12/\text{hour}$

(a) Expected number of units in the queue,

$$L_q= \lambda^2/[\mu.(\mu-\lambda)] = (6)^2/[12. (12-6)] = 0.5 \text{ units}$$

(b) Expected waiting time in the queue,

$$W_q = \lambda/[\mu.(\mu-\lambda)] = (6)/[12. (12-6)] = 0.0833 \text{ hours}$$

(c) Expected number of units in the system,

$$L_s = \lambda/(\mu-\lambda) = 6/(12-6) = 1 \text{ unit}$$

(d) Expected waiting time in the system,

$$W_s = 1/(\mu-\lambda) = 1/(12-6) = 0.1667 \text{ hours}$$

(e) Expected fraction of the day that the phone will be in use,

$$\rho = \text{Busy period} = \lambda / \mu = 6/12 = 1/2$$

(f) Probability that the customer will have to wait,

$$P(W > 0) = 1 - P_0 = 1 - (1- \rho) = \rho = 1/2 = 0.5$$

**Example:** Customers come at a one-man barbershop in a Poisson distribution, with a mean arrival rate of four per hour. If the clients are always willing to wait, the hair cutting time is exponentially distributed, with an average hair cut lasting 10 minutes. find:

(a) Average number of customer in the shop.
(b) Average waiting time of a customer.
(c) The probability that a customer will have to wait.
(d) Expected waiting time of customer who has to wait.

**Solution:** Given,
The mean arrival rate, $\lambda = 4$/hour
The mean service time = 10 min
The mean service rate, $\mu = (1/10) \times 60 = 6$/hour

(a) Average number of customer in the shop,

$$L_s = \lambda/(\mu-\lambda) = 4/(6-4) = 2 \text{ Customers}$$

(b) Average waiting time of a customer,

$$W_q = \lambda/[\mu.(\mu-\lambda)] = (4)/[6. (6-4)] = 0.333 \text{ hours}$$

(c) The probability that a customer will have to wait,

$$P(W > 0) = 1 - P_0 = 1 - (1- \rho) = \rho = \lambda / \mu = 4/6 = 0.667$$

(d) $(W/W > 0) = 1/(\mu-\lambda) = 1/(6-4) = 1/2$ hours = 30 mins

**Example:** A TV repairman works on the sets in the order that they are delivered and anticipates that each set's repair time will be exponentially dispersed, with a mean of 30 minutes. In a Poisson fashion, the sets come on average every 12 to 10 hours throughout the day. Determine:

(a) What is the expected idle time per day for the repairman?

(b) How many TV sets will be there waiting for the repair?

**Solution:** Given,

The mean service time = 30 mins

The mean arrival rate, $\lambda$ = 12/10 hours a day = 12/(10 x 60) = 1/50 per min

The mean service rate, $\mu$ = 1/30 per min

(a) Busy period, $\rho = \lambda / \mu = (1/50)/(1/30) = 0.6$ hour

The idle time, $P_0 = 1 - \rho = 1 - 0.6 = 0.4$ hour

The expected idle time per day for the repairman = 0.4 x 10 = 4 hrs/day

(b) The number of TV sets waiting for the repair,

$L_q = \lambda^2/[\mu.(\mu-\lambda)] = (1/50)^2/[(1/30).((1/30)-(1/50))] = 0.9$ units

**Example:**Trains arrive at a pace of 30 per day in a railway marshalling yard. Considering that the service time has an average of 36 minutes and that the inter-arrival time follows an exponential distribution. Calculate:

(a) The average number of trains in the system

(b) Expected length of non-empty queue

(c) The probability that the queue size exceeds 12

**Solution:** Given,

The mean arrival rate, $\lambda$ = 30 trains per day = 30/(24x60) = 1/48 trains per min

The mean service time = 36 mins

The mean service rate, $\mu$ = 1/36 trains per min

(a) The average number of trains in the system,

$L_s = \lambda/(\mu-\lambda) = (1/48)/((1/36)-(1/48)) = 3$ trains

(b) Expected length of non-empty queue,

$(L/L > 0) = \mu/(\mu-\lambda) = (1/36)/((1/36)-(1/48)) = 4$ trains

(c) The probability that the queue size exceeds 12,

$P(\text{Queue size} \geq 12) = (\lambda/\mu)^{12} = [(1/48)/(1/36)]^{12} = (0.75)^{12} = 0.032$

**IN-TEXT QUESTIONS**

5. ___________ is the average time that a customer spends in the system from the entry in the queue to the completion of the service.
6. The average number of customers in the queue waiting to receive the service is called ______________.
7. The expected number of services completed in the time interval of length unity is called:
   (a) Mean servicing rate
   (b) Mean arrival rate
   (c) Mean waiting time
   (d) Mean servicing time
8. The time taken for servicing of a unit is known as its _______________.

### 3.12.2 MODEL - II [(M/M/c):(∞/FCFS)]:

Assumptions for this model are as follow:
(i) The arrival of customer follows the Poisson distribution.
(ii) The service time follows the exponential distribution.
(iii) Several servers in the service facility are available.
(iv) The length of the queue is infinite.
(v) Customers are served on the basis of first come first serve service mechanism.
(vi) c = number of servers
(vii) Arrival rate, $\lambda = \lambda_n$ (Depending upon n)
(viii) Service rate, $\mu = \mu_n$ (Depending upon n)
(ix) $\mu_n = \begin{cases} n\mu, & \text{if } n \leq c \\ c\mu, & \text{if } n \geq c \end{cases}$

❖ To obtain the system of steady state equations

Consider similar arguments as in equations (1) and (2) of Model – I,

$P_n'(t) = \lambda.\ P_{n-1}(t) - (\lambda+n\mu).\ P_n(t) + (n+1)\mu.\ P_{n+1}(t)$, for $0 < n < c$ ……….(1)

$P_n'(t) = \lambda.\ P_{n-1}(t) - (\lambda+c\mu).\ P_n(t) + c\mu.\ P_{n+1}(t)$, for $n \geq c$ ……….(2)

$P_0'(t) = -\lambda.\ P_0(t) + \mu.\ P_1(t)$, for $n = 0$ ……….(3)

In steady state, the probabilities are independent of time, therefore equations (1), (2) & (3) becomes

$\lambda. P_{n-1} - (\lambda+n\mu). P_n + (n+1)\mu. P_{n+1} = 0$ ..........(4)

$\lambda. P_{n-1} - (\lambda+c\mu). P_n + c\mu. P_{n+1} = 0$ ..........(5)

$-\lambda. P_0 + \mu. P_1 = 0$ ...........(6)

These equations are called steady state equations.

❖ Now to solve the system of steady state equations

From equation (6), we have

$$P_1 = (\lambda/\mu). P_0 \quad \text{..........(7)}$$

Put n = 1 in equation (4) and using value of $P_1$, we get

$$P_2 = (\lambda/(2\mu)). P_1 = (1/2!).(\lambda/\mu)^2. P_0$$

Put n = 2 in equation (4) and using value of $P_1$, we get

$$P_3 = (\lambda/(3\mu)). P_2 = (1/3!).(\lambda/\mu)^3. P_0$$

----------------------------------------

$$P_n = (\lambda/(n\mu)). P_{n-1} = (1/n!).(\lambda/\mu)^n. P_0 \;, \quad n \le c$$

----------------------------------------

$$P_c = (\lambda/(c\mu)). P_{c-1} = (1/c!).(\lambda/\mu)^c. P_0$$

$$P_{c+1} = (\lambda/(c\mu)). P_c = (1/c).(1/c!).(\lambda/\mu)^{c+1}. P_0$$

$$P_{c+2} = (\lambda/(c\mu)). P_{c+1} = (1/c^2).(1/c!).(\lambda/\mu)^{c+2}. P_0$$

----------------------------------------

$$P_n = P_{c+(n-c)} = (1/c^{n-c}).(1/c!).(\lambda/\mu)^n. P_0 \;, \quad n \ge c$$

Now, in order to find $P_0$, use the fact that the total probability, $\sum_{n=0}^{\infty} P_n = 1$

$$\sum_{n=0}^{c-1} P_n + \sum_{n=c}^{\infty} P_n = 1$$

$$\sum_{n=0}^{c-1} [(1/n!).(\lambda/\mu)^n. P_0] + \sum_{n=c}^{\infty} [(1/c^{n-c}).(1/c!).(\lambda/\mu)^n. P_0] = 1$$

$$P_0 \Big[ \sum_{n=0}^{c-1} (c^n/n!).(\lambda/(c\mu))^n + \sum_{n=c}^{\infty} (c^c/c!).(\lambda/c\mu)^n \Big] = 1$$

$$P_0 \Big[ \sum_{n=0}^{c-1} (1/n!).(c\rho)^n + (c^c/c!) \sum_{n=c}^{\infty} (\rho)^n \Big] = 1 \qquad \text{(Since, } \rho = \lambda/c\mu\text{)}$$

$$P_0 \Big[ \sum_{n=0}^{c-1} (1/n!).(c\rho)^n + (c^c/c!).(\rho^c + \rho^{c+1} + \rho^{c+2} + \text{.....upto } \infty) \Big] = 1$$

$$_{n=0}$$

$$P_0 \left[ \sum_{n=0}^{c-1} (1/n!).(c\rho)^n + (c^c/c!).(\rho^c/(1-\rho)) \right] = 1$$

$$P_0 = \left[ \sum_{n=0}^{c-1} (1/n!).(c\rho)^n + (c^c/c!).(\rho^c/(1-\rho)) \right]^{-1}$$

**Important formulae of Model - II**

- $L_q = (c^c/c!).[\rho^{c+1}/(1-\rho)^2] = P_c.[\rho/(1-\rho)^2]$
- $L_s = L_q + \lambda/\mu$
- $(L/L > 0) = 1/(1-\rho)$
- $W_q = L_q/\lambda = P_c/[c\mu(1-\rho)^2]$
- $W_s = L_s/\lambda = W_q + (1/\mu)$
- $(W/W > 0) = 1/(c\mu-\lambda)$
- $\rho$ = Busy period = $\lambda/(c\mu)$
- $P_0 = \left[ \sum_{n=0}^{c-1} (1/n!).(c\rho)^n + (c^c/c!).(\rho^c/(1-\rho)) \right]^{-1}$
- $P_n = (1/n!).(\lambda/\mu)^n.P_0 \quad , \quad n \le c$
- $P_n = (1/c^{n-c}).(1/c!).(\lambda/\mu)^n.P_0, \quad n \ge c$
- $P(W > 0) = P_c/(1-\rho)$

**Example:**There are two long distance providers for a telephone exchange. The telephone company discovers that long distance calls typically come at a rate of 15/hour during peak load, as predicted by the poisson distribution. These conversations' durations are roughly exponentially distributed, with a mean duration of 5 minutes. Find:

(a) How likely is it that a subscriber will have to wait for long distance calls during the busiest time of the day?

(b) What is the average waiting time for the customers?

**Solution:** Given,

Number of servers, c =2

The mean arrival rate, $\lambda$ = 15 per hour

The mean service time = 5 min

The mean service rate, $\mu$ = 1/5 per min = 60/5 per hour = 12 per hour

Now, $\rho = \lambda/(c\mu) = 15/(2x12) = 5/8$

$$P_0 = \left[ \sum_{n=0}^{c-1} (1/n!).(c\rho)^n + (c^c/c!).\ (\rho^c/(1-\rho)) \right]^{-1}$$

$$P_0 = \left[ \sum_{n=0}^{1} (1/n!).(5/4)^n + (2^2/2!).\ ((5/8)^2/(1-(5/8))) \right]^{-1}$$

$$P_0 = [1 + (5/4) + (5/4)^2 .(4/3)]^{-1} = 3/13$$

(a) $P(W > 0) = P_c/(1-\rho) = [(1/c!).(\lambda/\mu)^c.\ P_0\,]/(1-\rho)$
$= [(1/2!).(15/12)^2.\ (3/13)]/(1-(5/8)) = 0.48$

(b) $W_q = P_c\,/[c\mu(1-\rho)^2] = [(1/c!).(\lambda/\mu)^c.\ P_0\,]\,/[c\mu(1-\rho)^2]$
$= [(1/2!).(15/12)^2.\ (3/13)]\,/[2x12.\ (3/8)^2] = 0.053$ hours $= 3.2$ mins

**Example:** A tax consulting firm has 3 counters in its offices to receive the people who have problems concerning their income and the sales tax. On an average 48 persons arrive in 8hours a day. Each tax advisor spends 15 min on an average for an arrival if the arrival time follows a Poisson distribution and the service time follows an exponential distribution. Find:

(a) The typical user count in the system.
(b) The customer's system-wide average wait period.
(c) The typical amount of customers in line for service.
(d) The length of time that consumers typically wait in line.
(e) The likelihood that a customer will need to wait before receiving assistance.

**Solution:** Given,
Number of servers, $c = 3$
The mean arrival rate, $\lambda = 48$ persons 8 hours a day $= 48/8 = 6$ / hour
The mean service time = 15 min
The mean service rate, $\mu = 1/15$ per min $= 60/15$ per hour $= 4$ / hour
Now, $\rho = \lambda\,/(c\mu) = 6/(3x4) = 1/2$

$$P_0 = \left[ \sum_{n=0}^{c-1} (1/n!).(c\rho)^n + (c^c/c!).\ (\rho^c/(1-\rho)) \right]^{-1}$$

$$P_0 = \left[ \sum_{n=0}^{2} (1/n!).(3/2)^n + (3^3/3!).\ ((1/2)^3/(1-(1/2))) \right]^{-1}$$

$P_0 = [1 + (3/2) + (1/2!).(3/2)^2 + 2.(1/3!).(3/2)^3]^{-1} = 0.21$

(a) The average number of customer in the system,

$$L_s = L_q + \lambda/\mu = (c^c/c!).[\rho^{c+1}/(1-\rho)^2] + \lambda/\mu$$
$$= (3^3/3!).[(1/2)^4/(1-(1/2))^2] + 6/4 = 1.73 \text{ customers}$$

(b) Average waiting time of the customer in the system,

$$W_s = L_q/\lambda + (1/\mu) = (c^c/c!).[\rho^{c+1}/(1-\rho)^2]/\lambda + (1/\mu)$$
$$= (3^3/3!).[(1/2)^4/(1-(1/2))^2]/6 + (1/4) = 0.289 \text{ hrs}$$

(c) Average number of customers waiting in the queue for service,

$$L_q = (c^c/c!).[\rho^{c+1}/(1-\rho)^2] = (3^3/3!).[(1/2)^4/(1-(1/2))^2] = 0.23$$

(d) Average waiting time of the customers in the queue,

$$W_q = L_q/\lambda = 0.23/6 = 0.038$$

(e) Probability that a customer has to wait before he gets service

$$= 1 - P_0 - P_1 - P_2$$
$$= 1 - P_0 - (\lambda/\mu).P_0 - (1/2!).(\lambda/\mu)^2.P_0$$
$$= 1 - 0.21 - (6/4).(0.21) - (1/2!).(6/4)^2.(0.21)$$
$$= 0.239$$

### 3.12.3 MODEL - III [(M/M/1):(N/FCFS)]:

Assumptions for this model are as follow:

(i) The arrival of customer follows the Poisson distribution.

(ii) The service time follows the exponential distribution.

(iii) There is only one queue and one server available.

(iv) The length of the queue is finite (say N).

(v) Customers are served on the basis of first come first serve service mechanism.

(vi) Arrival rate, $\lambda = \lambda_n$

(vii) Service rate, $\mu = \mu_n$

(viii) $\lambda_n = \begin{cases} \lambda, & \text{if } n < N \\ 0, & \text{if } n \geq N \end{cases}$

(ix) $\rho = \lambda/\mu$

❖ To obtain the system of steady state equations

Consider similar arguments as in equations (1) and (2) of Model – I, we can write

$P_n'(t) = \lambda.P_{n-1}(t) - (\lambda+\mu).P_n(t) + \mu.P_{n+1}(t)$ , for $0 < n < N$ ……….(1)

$P_N'(t) = \lambda.\ P_{N-1}(t) - \mu.\ P_N(t)$ , for n = N ..........(2)

$P_0'(t) = -\lambda.\ P_0(t) + \mu.\ P_1(t)$ , for n = 0 ..........(3)

In steady state, the probabilities are independent of time, therefore equations (1), (2) & (3) becomes

$\lambda.\ P_{n-1} - (\lambda+\mu).\ P_n + \mu.\ P_{n+1} = 0$ , for $0 < n < N$ ...........(4)

$\lambda.\ P_{N-1} - \mu.\ P_N = 0$ , for n = N ...........(5)

$-\lambda.\ P_0 + \mu.\ P_1 = 0$ , for n = 0 ...........(6)

These equations are called steady state equations.

❖ Now to solve the system of steady state equations

from equation (6)

$$P_1 = (\lambda/\mu).\ P_0 = \rho.\ P_0 \qquad ..........(7)$$

Put n = 1 in equation (4) and using value of $P_1$, we get

$$P_2 = (\lambda/\mu)^2.\ P_0 = \rho^2.\ P_0$$

Put n = 2 in equation (4) and using value of $P_1$, we get

$$P_3 = (\lambda/\mu)^3.\ P_0 = \rho^3.\ P_0$$

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$$P_n = (\lambda/\mu)^n.\ P_0 = \rho^n.\ P_0 \quad , \text{for } n < N$$

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$$P_N = (\lambda/\mu)^N.\ P_0 = \rho^N.\ P_0 \quad , \text{for } n = N$$

$$P_{N+1} = 0 \quad , \text{for } n > N$$

Now, in order to find $P_0$, use the fact that the total probability, $\sum_{n=0}^{N} P_n = 1$

$$P_0 + P_1 + P_2 + \ldots\ldots + P_N = 1$$

$$P_0 + \rho.\ P_0 + \rho^2.\ P_0 + \ldots\ldots. + \rho^N.\ P_0 = 1$$

$$P_0\ [1 + \rho + \rho^2 + \ldots\ldots. + \rho^N] = 1$$

$$P_0\ [(1-\rho^{N+1})/(1-\rho)] = 1$$

$$P_0 = (1-\rho)/\ (1-\rho^{N+1})$$

Hence,

$$P_n = \rho^n.\ P_0 = \rho^n.\ [(1-\rho)/\ (1-\rho^{N+1})] \quad , \text{for } n \le N$$

**Important formulae of Model - III**

- $P_0 = (1-\rho)/(1-\rho^{N+1})$
- $P_n = \rho^n . [(1-\rho)/(1-\rho^{N+1})]$ , for $n \leq N$
- $L_s = P_0 . \sum_{n=0}^{N} (n.\rho^n)$
- $L_q = P_0 . \sum_{n=1}^{N} (n-1).\rho^n$
- $W_s = L_s / \lambda$
- $W_q = L_q / \lambda = W_s - (1/\mu)$
- $\rho = \lambda/\mu$

**Example:**Think about a single server queuing system with exponential response time and poisson arrival. Five customers come per hour, and services last 30 minutes. if the device can only accommodate four users. Find:

(a) The likelihood that the system is vacant.

(b) The typical user count in the system.

(c) The typical amount of customers waiting in line.

**Solution:** Given,

The mean arrival rate, $\lambda = 5$ / hour

The mean service time = 30 min

The mean service rate, $\mu = 1/30$ per min = 60/30 per hour = 2 / hour

N = 4

Now, $\rho = \lambda / \mu = 5/2 = 2.5$

(a) Probability that the system is empty,

$$P_0 = (1-\rho)/(1-\rho^{N+1}) = (1-2.5)/(1-(2.5)^5) = 0.016$$

(b) The average number of customers in the system,

$$L_s = P_0 . \sum_{n=0}^{4} (n.\rho^n) = P_0 . [0.\rho^0 + 1.\rho^1 + 2.\rho^2 + 3.\rho^3 + 4.\rho^4]$$

$$= 0.016 [0 + (2.5) + 2(2.5)^2 + 3(2.5)^3 + 4(2.5)^4]$$

$$= 0.016 [0 + (2.5) + 2(6.25) + 3(15.625) + 4(39.0625)]$$

$$= 0.016 [0 + 2.5 + 12.5 + 46.875 + 156.25]$$

$$= 3.49$$

(c) Average number of customers in the queue,

$$L_q = P_0 . \sum_{n=1}^{4} (n-1).\rho^n = P_0 . [0.\rho^1 + 1.\rho^2 + 2.\rho^3 + 3.\rho^4]$$

$= 0.016 [0(2.5)+ 1(2.5)^2 + 2(2.5)^3 +3(2.5)^4]$
$= 0.016 [0 + 1(6.25) + 2(15.625) +3(39.0625)]$
$= 0.016 [0 + 6.25 + 31.25 + 117.1875]$
$= 2.475$

**Example:** Three clients can be served at once in a one-person barbershop, two of whom can wait while the other is being attended to. If a customer arrives and the store is closed, he moves to another store. The average rate of random customer arrival is 4 per hour, and the average service duration is 10 minutes. Determine:

(a) The probability distribution for how many customers are in line for assistance.
(b) The amount of patrons anticipated to be waiting in the store.
(c) The anticipated clientele at the barbershop.
(d) How long can a client anticipate being in the store for?

**Solution:** Given,
The mean arrival rate, $\lambda = 4$ / hour
The mean service time = 10 min
The mean service rate, $\mu = 1/10$ per min = 60/10 per hour = 6 / hour
N = 3
Now, $\rho = \lambda / \mu = 4/6 = 2/3 = 0.667$

(a) The probability distribution for the number of customers waiting for service,

$P_0 = (1-\rho)/ (1- \rho^{N+1}) = (1-0.667)/ (1- (0.667)^4) = 0.4152$
$P_1 = \rho. P_0 = 0.667 \times 0.4152 = 0.2769$
$P_2 = \rho^2. P_0 = (0.667)^2 \times 0.4152 = 0.1847$
$P_3 = \rho^3. P_0 = (0.667)^3 \times 0.4152 = 0.1232$

(b) Expected number of customers waiting in the shop,

$$L_q = P_0 . \sum_{n=1}^{3} (n-1).\rho^n = P_0 . [0.\rho^1 + 1.\rho^2 + 2.\rho^3]$$

$= 0.4152 [0(0.667)+ 1(0.667)^2 + 2(0.667)^3]$
$= 0.4152 [0 + 0.44489 + 2(0.29674)]$
$= 0.4311$

(c) Expected number of customers in the barber's shop.

$$L_s = P_0 . \sum_{n=0}^{3} (n.\rho^n) = P_0 . [0.\rho^0 + 1.\rho^1 + 2.\rho^2 + 3.\rho^3]$$

$$= 0.4152\ [0 + (0.667) + 2(0.667)^2 + 3(0.667)^3]$$
$$= 0.4152\ [0 + (0.667) + 2(0.44489) + 3(0.29674)]$$
$$= 1.016$$

(d) The time that a customer can expect to spend in the shop,

$W_s = L_s /\lambda = (1.016/4)$ hrs = 15.24 min

### 3.12.4 MODEL - IV [(M/M/c):(N/FCFS)]:

Assumptions for this model are as follow:

(i) The arrival of customer follows the Poisson distribution.
(ii) The service time follows the exponential distribution.
(iii) Several servers in the service facility are available.
(iv) The length of the queue is finite (say N).
(v) Customers are served on the basis of first come first serve service mechanism.
(vi) c = number of servers
(vii) Arrival rate, $\lambda = \lambda_n$
(viii) Service rate, $\mu = \mu_n$
(ix) $\lambda_n = \begin{cases} \lambda, & \text{if } 0 \le n \le N \\ 0, & \text{if } n > N \end{cases}$
(x) $\mu_n = \begin{cases} n\mu, & \text{if } 0 \le n \le c \\ c\mu, & \text{if } c \le n \le N \end{cases}$
(xi) $\rho = \lambda /(c\mu)$

❖ To obtain the system of steady state equations

Consider similar arguments as in equations (1) and (2) of Model – I,

$P_n'(t) = \lambda. P_{n-1}(t) – (\lambda+n\mu). P_n(t) + (n+1)\mu. P_{n+1}(t)$, for $0 \le n < c$ ……….(1)

$P_n'(t) = \lambda. P_{n-1}(t) – (\lambda+c\mu). P_n(t) + c\mu. P_{n+1}(t)$, for $c \le n \le N$ ……….(2)

$P_0'(t) = - \lambda. P_0(t) + \mu. P_1(t)$, for $n = 0$ ……….(3)

In steady state, the probabilities are independent of time, therefore equations (1), (2) & (3) becomes

$\lambda. P_{n-1} – (\lambda+n\mu). P_n + (n+1)\mu. P_{n+1} = 0$ ……….(4)

$\lambda. P_{n-1} – (\lambda+c\mu). P_n + c\mu. P_{n+1} = 0$ ……….(5)

$- \lambda. P_0 + \mu. P_1 = 0$ ………..(6)

These equations are called steady state equations.

❖ Now to solve the system of steady state equations

From equation (6), we have

$$P_1 = (\lambda/\mu). P_0 = (\rho c). P_0 \qquad \text{..........(7)}$$

Put n = 1 in equation (4) and using value of $P_1$, we get

$$P_2 = (\lambda/(2\mu)). P_1 = (1/2!).(\lambda/\mu)^2. P_0 = (1/2!).(\rho c)^2. P_0$$

Put n = 2 in equation (4) and using value of $P_1$, we get

$$P_3 = (\lambda/(3\mu)). P_2 = (1/3!).(\lambda/\mu)^3. P_0 = (1/3!).(\rho c)^3. P_0$$

-----------------------------------------------------------

-----------------------------------------------------------

$$P_n = (\lambda/(n\mu)). P_{n-1} = (1/n!).(\lambda/\mu)^n. P_0 = (1/n!).(\rho c)^n. P_0 \ , 0 \le n \le c$$

-----------------------------------------------------------

-----------------------------------------------------------

$$P_c = (\lambda/(c\mu)). P_{c-1} = (1/c!).(\rho c)^c. P_0$$

$$P_{c+1} = (\lambda/(c\mu)). P_c = (1/c).(1/c!).(\rho c)^{c+1}. P_0$$

$$P_{c+2} = (\lambda/(c\mu)). P_{c+1} = (1/c^2).(1/c!).(\rho c)^{c+2}. P_0$$

-----------------------------------------------------------

-----------------------------------------------------------

$$P_n = P_{c+(n-c)} = (1/c^{n-c}).(1/c!).(\rho c)^n. P_0 = (c^c/c!).\rho^n. P_0, \ \ c \le n \le N$$

Since the capacity of the system is N, therefore

$$P_n = 0 \text{ , for } n > N$$

Now, in order to find $P_0$, use the fact that the total probability, $\sum_{n=0}^{\infty} P_n = 1$

$$\sum_{n=0}^{c-1} P_n + \sum_{n=c}^{N} P_n = 1$$

$$\sum_{n=0}^{c-1} [(1/n!).(\rho c)^n. P_0] + \sum_{n=c}^{N} [(c^c/c!).\rho^n. P_0] = 1$$

$$P_0 \left[ \sum_{n=0}^{c-1} (1/n!).(\rho c)^n + \sum_{n=c}^{N} (\rho c)^c.(1/c).(\rho)^{n-c} \right] = 1$$

$$P_0 \left[ \sum_{n=0}^{c-1} (1/n!).(\rho c)^n + (\rho c)^c.(1/c!) \sum_{n=c}^{N} (\rho)^{n-c} \right] = 1 \ \text{(Since, } \rho = \lambda/c\mu)$$

$$P_0 \left[ \sum_{n=0}^{c-1} (1/n!).(\rho c)^n + (\rho c)^c.(1/c!).(1+\rho^1+\rho^2+\ldots..+\rho^{N-c}) \right] = 1$$

$$P_0 \left[ \sum_{n=0}^{c-1} (1/n!).(\rho c)^n + (\rho c)^c.(1/c!).\{(1-\rho^{N-c+1})/(1-\rho)\} \right] = 1$$

$$P_0 = \left[ \sum_{n=0}^{c-1} (1/n!).(\rho c)^n + (\rho c)^c.(1/c!).\{(1-\rho^{N-c+1})/(1-\rho)\} \right]^{-1} \quad ,(\text{if } \rho = \lambda/(c\mu) \neq 1)$$

$$P_0 = \left[ \sum_{n=0}^{c-1} (1/n!).(\rho c)^n + (\rho c)^c.(1/c!).(N-c+1) \right]^{-1} \quad ,(\text{if } \rho = \lambda/(c\mu) = 1)$$

**Important formulae of Model - IV**

- $L_q = \sum_{n=sc}^{N} [(n-c).P_n]$
- $L_s = \sum_{n=0}^{N} [n.P_n]$
- $W_q = L_q/\lambda'$
  (where λ' is the effective arrival rate and is given by, $\lambda' = \lambda(1-P_N)$)
- $W_s = L_s/\lambda'$
- $\rho = \lambda/(c\mu)$
- $$P_0 = \begin{cases} \left[ \sum_{n=0}^{c-1} (1/n!).(\rho c)^n + (\rho c)^c.(1/c!).\{(1-\rho^{N-c+1})/(1-\rho)\} \right]^{-1} & ,(\text{if } \rho = \lambda/(c\mu) \neq 1) \\ \left[ \sum_{n=0}^{c-1} (1/n!).(\rho c)^n + (\rho c)^c.(1/c!).(N-c+1) \right]^{-1} & ,(\text{if } \rho = \lambda/(c\mu) = 1) \end{cases}$$
- $$P_n = \begin{cases} (1/n!).(\rho c)^n.P_0 & ,\ 0 \le n \le c \\ (c^c/c!).\rho^n.P_0 & ,\ c \le n \le N \\ 0 & ,\ \text{for } n > N \end{cases}$$

**Example:** Allow for the establishment of a car inspection station with 3 examination stalls. Assume that a vehicle waits so that it can move to the front of the line when a stall becomes available. The station has room for nearly 4 vehicles to wait at once. The station can only hold 7 vehicles at a time. During peak hours, a mean of one vehicle arrives according to the poisson distribution per minute. With a mean of six minutes, the service duration follows an exponential distribution. Determine:

(a) The typical amount of cars waiting in line.
(b) The typical amount of vehicles using the system at peak times.
(c) The system's anticipated wait period.
(d) The anticipated volume of vehicles per hour that are unable to access the station.

**Solution:** Given, c = 3, N = 7

Mean arrival rate, $\lambda = 1$ car per min

Mean service time = 6 min

Mean servicing rate, $\mu = 1/6$ car per min

$\rho = \lambda/(c\mu) = 1/(3(1/6)) = 2$

$$P_0 = [\sum_{n=0}^{c-1} (\rho c)^n/n! + ((\rho c)^c/c!)\,[(1-\rho^{N-c+1})/(1-\rho)]^{-1}$$

$$= [\sum_{n=0}^{c-1} (6)^n/n! + ((6)^3/3!).(1-2^{7-3+1})/(1-2)]^{-1}$$

$$= [1 + 6 + (6^2/2!) + 1116]^{-1} = 1/1141$$

(a) The average number of cars in the queue,

$$L_q = \sum_{n=c}^{N} (n-c)P_n$$

$$L_q = \sum_{n=3}^{7} (n-3)(c^c/c!)\,\rho^n\, P_0$$

$$L_q = \sum_{n=3}^{7} (n-3)(3^3/3!)\,(2)^n\, P_0$$

$$= (1/1141)[0 + (3^3/3!)2^4 + 2(3^3/3!)2^5 + 3(3^3/3!)2^6 + 4(3^3/3!)2^7]$$

$$= [72 + 288 + 864 + 2304] = 3.09$$

(b) The average number of cars in the system during peak hours,

$$L_s = \sum_{n=0}^{N} n.P_n$$

$$L_s = \sum_{n=0}^{c} n.P_n + \sum_{n=c+1}^{N} n.P_n$$

$$L_s = \sum_{n=0}^{3} n.[(1/n!)\,(\rho c)^n\, P_0\,] + \sum_{n=4}^{7} n.[(c^c/c!)\,\rho^n\, P_0]$$

$$L_s = [\sum_{n=0}^{3} n.(1/n!)\,(6)^n + \sum_{n=4}^{7} n.(c^c/c!)\,\rho^n]\, P_0$$

$L_s = [6 + 6^2 + (6^3/2) + 4(3^3/3!)2^4 + 5(3^3/3!)2^5 + 6(3^3/3!)2^6 + 7(3^3/3!)2^7].\,(1/1141)$

$= [6 + 36 + 108 + 288 + 720 + 1728 + 4032].(1/1141) = 6.06$

(c) The expected waiting time in the system,

$W_s = L_s/(\lambda(1-P_N))$

$= 6.06/\,(1-(c^c/c!)\,\rho^N\, P_0)$

$= 6.06/\,[1-(3^3/3!)\,2^7\,(1/1141)]$

$= 12.23$ min

(d) The expected number of cars per minute that cannot enter in the station

= Arrival rate x Probability that the system is full

= Arrival rate x Probability that there are N units in the system

$= \lambda.P_N$

Hence, the expected number of cars per hour that cannot enter in the station

$= 60.\lambda.P_N$

$= 60.(1).P_7$

$= 60.\,(c^c/c!)\,\rho^7\, P_0$

$= 60.\,(3^3/3!)\,2^7.\,(1/1141)$

= 30.3 cars per hour

## 3.13 APPLICATIONS OF QUEUING THEORY

Many different business scenarios have been used to apply the waiting line or queuing theory. There are likely to be queues waiting in any circumstance where there are clients, including banks, post offices, movie theatres, gas stations, train ticket desks, doctor's offices, etc. Customers typically want a specific degree of service, whereas businesses that provide service facilities work to keep costs down while still providing the required service. Queuing theory can be used to solve issues like the ones listed below:

- Aircraft scheduling at crowded airports for takeoff and landing.
- Planning the distribution and collection of tools by employees from tool cribs in factories.
- Fleet scheduling for mechanical transport.
- Scheduling of assembly line parts and components.
- Controlling and analysing inventories.
- Minimization of congestion due to traffic delay at tool booths.
- Routing and scheduling of salesman and sales efforts.
- Provide models that are capable of influencing arrival pattern of customers or determine the most appropriate amount of service or number of service stations.

## 3.14 LIMITATIONS OF QUEUING THEORY

The traditional queuing theory's presumptions might be too severe to accurately simulate practical scenarios. These models are unable to handle the complexity of production lines with product-specific characteristics. To simulate, evaluate, visualise, and optimise time dynamic queuing line behaviour, specific tools have been developed. Following are a few of queueing theory's drawbacks:

- The majority of queuing models are highly complicated and difficult to comprehend.
- The exact theoretical distribution that would apply to a particular queue situation is frequently unknown.
- The analysis of waiting problems becomes more challenging if the queuing discipline does not follow the "first come, first serve" principle.

## 3.15 SUMMARY

- The queuing model's goal is to determine the best service rate and server count in order to reduce both the typical cost of using the system and the cost of providing service.
- If arrivals are entirely random, a poisson's distribution can be used to describe the chance distribution of the number of arrivals over a given period of time.

- When a system's working characteristics rely on time, it is said to be in a transient state, and when they stop being dependent on time, it is said to be in a steady state.
- If the system's arrival rate exceeds its servicing rate, the queue's length will grow over time and eventually reach infinite as time tends towards infinity. An explosive condition is one in which this occurs.
- David G. Kendall proposed the first notation for a queuing model's features in 1953.

## 3.16 GLOSSARY

**Waiting Lines** - Queues or waiting lines are a typical occurrence in both regular life and several corporate and industrial settings.

**Input source of queue** - Customers requiring service are generated at different times by an input source, commonly known as population.

**Queue discipline (Service discipline)** - The queue discipline is the order or manner in which customers from the queue are selected for service.

**System output** - The rate at which consumers are served is referred to as system output. It depends on how long the facility needs to provide the service and how the service facility is set up.

**Transient state** - When a system's operational features rely on time, it is said to be in a transitory state.

**Steady state** - A queuing system is said to be in a steady state when the probability of having a certain amount of customers is independent of time.

**Explosive state** - The length of the queue will grow over time and eventually reach infinity if the system's arrival rate is higher than its service rate. Such a state is known as explosive state.

## 3.17 ANSWERS TO IN-TEXT QUESTIONS

| | |
|---|---|
| 1. Optimum<br>2. All of above<br>3. Balking<br>4. Poisson distribution | 5. Waiting time in the system<br>6. Queue length<br>7. Mean servicing rate<br>8. Servicing time |

## 3.18 SELF-ASSESSMENT QUESTIONS

1. What do you mean by queue? Describe the basic elements of queues.
2. Explain different states of queuing system.
3. What are the applications of queuing theory?
4. Define the role of poisson and exponential distribution in queuing theory.
5. Customers appear at a particular gas station at a poisson distribution-based average rate of 12 per hour. The service duration is distributed exponentially, with a mean of 2 minutes. Then find:
    a) Traffic intensity
    b) Average length of the queue.
    c) The expected number of customer at petrol pump.
6. Assume that customers appear at a bank cashier window at a poisson-distributed average rate of 20 per hour. 30 clients are served by the bank cashier in an hour. There is no cap on the length of the potential queue, and the customers who arrive from an infinite population are served first.
    a) What is the value of utilization factor?
    b) What is the expected waiting time in the system per customer?
    c) What is the probability of zero customers in the system?
7. A poison-distributed entry rate of 25 people per hour is observed at a movie theatre ticket counter. With an average service rate of two minutes per service, the distribution is exponential. Calculate the average number of people in the waiting queue, the average length of the line, the average duration of the system, and the utilisation factor?

8. The machines in production shop breakdown at an average of 2 per hour. The non productive time of any machine costs Rs.30 per hour. If the cost of repairman is Rs.50 per hour and the service rate is 3 per hour. Determine:
   a) The number of machines not working at any point of time.
   b) The average time that a machine is waiting for the repairman.
   c) The cost of non-productive time of the machine per hour.
   d) The expected cost of system per hour.
9. Patrons arrive at a reception counter at the rate of 2 per min. The receptionist in duty takes an average of 1 min per patron. Calculate:
   a) What is the chance that a patron will straight way meet the receptionist?
   b) The probability that the receptionist is busy.
   c) Average number of patrons in the system.
10. In a car manufacturing plant, a loading crane takes exactly 10 min to load a car into a wagon and again comes back to the position to load another car. If the arrival of cars is in a poisson stream at an average rate of one after every 20 min, calculate:
   a) The average waiting time of a car in the queue.
   b) The average waiting time in the system.

## 3.19 SUGGESTED READINGS


- Vohra N. D. (2021). Quantitative Techniques in Management.6th ed.,*Tata McGraw Hill*.
- Taha H. A. (2014). Operations Research– An Introduction. 9th ed., *Prentice Hall India*.

**LESSON 4**

**SIMULATION**

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## STRUCTURE

4.1. Learning Objectives

4.2. Introduction of Simulation

4.3. Key Advantages of Simulation for Business

4.4. General Elementary Steps in the Simulation Technique

4.5. Types Of Simulation Models to Control in Management Science

4.6 Monte Carlo Simulation

4.7. Tools For the Verification and Validation of Simulation Model

4.7.1. Methods To Execute Verification of Simulation Model

4.7.2. Methods To Execute Validation of Simulation Model

4.7.3. Model Data Comparison with Real Facts

4.7.3.1. Validating The Current System

4.7.3.2. Validating The First Time Model

4.8. Advantages And Limitations of Simulation

4.9. Summary

4.10. Self-Assessment Exercises

4.11. Objective Questions

4.12 Suggested Readings

## 4.1 LEARNING OBJECTIVES

**Simulation refers a explanatory form that admits a resolution creator to estimate the nature of a model under differing environments.**

It is well known that, not all real-world problems can be solved by applying a specific type of technique so-called mathematical models and formulas that could be applied to certain types of problems and then performing the calculations. Some problem situations are too complex to be represented by the concise techniques presented so far in this text. In such cases, simulation is an alternative form of analysis tool for computational solution of problems.

Simulation is a explanatory calculation system at which point a model of a process is create andbefore studies are attended on the model to describe allure conduct under various lifestyle. Unlike many of the added models characterized in the manual, it is not an optimizing finish. Simulation models promote resolution creators to interrogate definitely alternatives trough a what- if methods

The use of simulation as a decision-making tool is fairly widespread, and you are definitely familiar with some of the ways it is used. Other reasons for the demand of simulation contains:

- ✓ Many positions are also complex to permit growth of a mathematical resolution; the quality of interpretation wanted would seriously influence the results. In contrast, imitation models are frequently smart to capture the copiousness of a situation outside waive unity, through enhancing the conclusion process.
- ✓ Simulation models are justly natural to use and easy to understand.
- ✓ Simulation allows the conclusion creator to conduct experiments on a model that will help in understanding process action while preventing the risks of transporting tests on the model's real facts.
- ✓ General computer software packages handy it simple to use properly exclusive models.
- ✓ Simulation maybe alternative for a off-course range of positions.
- ✓ There have existed abundant successful uses of these methods.

This Simulation idea is best implicit accompanying an model: assume construction a model of by virtue of what a day at a general store pans out. You assemble rules for in what way or manner community will communicate, when equipment are brought, 'congestion,' 'free time,' and anything different to build an correct model of that real-world sells ground.

You will therefore run that model, usually in simulation program, to visualize the results of achieving those rules against variables, in the way that a late supply consignment or a Black Friday surge.

There are generally three positions in which you would be going to use simulation models:

- When you lack info, that is accepted when examining old or historical occurrences.
- When your trade processes are also complex expected resolved through usual orders.
- When you need to experiment in a cheap, low-risk atmosphere. ( For example, If you be going to implement a dangerous, high-priced change to your trade, and need to validate).

## 4.3 KEY ADVANTAGES OF SIMULATION FOR BUSINESS

Those positions perform self-explanatory, but skilled are deeper-level trade advantages to running a imitation model, that we investigate in detail beneath.

1. **Flexibility**
2. **Test Large and/or Complex Systems**
3. **Isolated from the real world Counterpart**
4. **Focus on Theoretical, "What-if" Queries**
5. **Study the Effects of Different, Interrelated (Consistent) Variables**
6. **Time density**
7. **Test of Complications**

- **Flexibility:**

✓ You can pretend many various things. From trade movements to preparation aircraft pilots, skilled is no deficiency of existent and potential applications for simulation structures.

✓ When it comes to simulate for trade (like, the retail instance determined above), you can engage it to capture insights in excavating, production, sell, supply chain management, management, and many remainder of something. It's manufacturing agnostic and appropriate to innumerable use-cases.

- **Test Large and/or Complex Systems:**

✓ If you have enough calculating capacity, you can simulate amazingly complex scenarios, in the way that the regular movements of an airstrip through an complete quarter or a city traffic gridiron.

✓ It doesn't matter how many rules you put or variables you confuse at bureaucracy. As long as you have the need calculating capacity, you can simulate it accompanying relative ease.

✓ Presently, simulating big atmospheres, in the way that airports, is the standard. The distinctness actually display or take public by virtue of what to model and resolve simulations, not the idea of imitation posing itself.

- **Isolated from the real world Counterpart:**

✓ With simulation modeling, you can generate copious amounts of insights without ever touching the real-world system.

✓ This is a meaningful benefit for big surroundings.

✓ Be it airports, excavating movements, all-encompassing transportation and distribution ventures, or airplane congregation, all of these are multi-billion-currency movements.

✓ Inserting even individual change in a complex, big process can influence delays and control of product quality questions command a price of tens or a great number of millions of greenbacks.

✓ With simulation, you can field-test your changes before they are executed in the here and now. You can take understandings about potential risks early, and act in advance of ruling class.

- Take, e.g., closing a road: agreed, it will cause a bottleneck, but accompanying simulation, you will see when that obstacle will be most harsh. You can understand road construction crews to not affiliate with organization the district all the while congestion (so concerning manage easier for traffic to flow).

- **Focus on Theoretical, "What-if" Queries**

- Whether it better understands old civilizations or construction-up business wit, you can again use simulation designing to produce insights for their own well-being.

- The info may be irrelevant contemporary, but it could be appropriate from now on when the right factors (like, technology, regulatory atmosphere, etc.,) occur.

- **Study the Effects of Different, Interrelated (Consistent) Variables**

- ✓ It's accepted for complex movements to involve many various determinants.

- ✓ In manufacturing, for example, you depend on hundreds or thousands of machines, a logistics chain, suppliers, access to materials, and human labor.

- ✓ In production, e.g., you depend on a great number or chiliads of machines, a management chain, suppliers, access to stocks, and human labor.

- ✓ With simulation modeling, you can acquire an understanding of by virtue of what your production movements will experience as a result of a changing, in the way that distressing weather, a worker strike, a governmental confrontation in a country that supplies natural resources, and so forth.

- ✓ This is valuable news for conclusion-makers, leaders, and shareholders the one are analyzing project proposals and changes to their existent wholes.

- **Time density**

- Though you want visions covering months or age into the future, you cannot afford to wait that long to receive ruling class. With simulation modeling, you can acquire facts about the complete for instance is 12 months and comparatively fast example is inside 1 daytime.

- For example, Texas A&M scientists currently fake the potential of biomass hike in cool seasons.

- **Test of Complications**

✓ Be it the inference of a new tool in a firm or a new departmental process, you can test to visualize if it everything as destined through simulation modeling.

✓ In addition, you can recognize potential difficulties and combine resolutions for those (and test repeated) before executing your change in the here and now.

✓ As you can visualize, simulation modeling determines a roomy range of trade benefits. If it may be summed up into individual plan, it hopeful that of impartially concluding the results of your conduct before they take place in the here and now. '***20:20' knowledge is not restricted to just retrospect***.

## 4.4 GENERAL ELEMENTRY STEPS IN THE SIMULATION TECHNIQUE

Regardless of the type of imitation complicated, following fundamental steps are used for all simulation models:

1. Identify the complications and set aims.
2. Develop the simulation model.
3. Test the model undoubtedly that it indicates bureaucracy being intentional.
4. Develop individual or more experiments (environments under that the model's conduct will be checked).
5. Run the simulation and judge the results.
6. Repeat steps 4 and 5 as far as you are compensated accompanying the results.

***The beginning task in problem solving (identification)*** of some sort search out simply acknowledge the problem and set goals that the answer is engaged to obtain; simulation is no omission. A clear announcement of the aims can determine not only managing for model designing but too the balance for estimation of the accomplishment or failure of a simulation. In general, the aim of a simulation study search out decide by virtue of what a arrangement will function under sure environments. The more distinguishing a organizer is about what he or she is expect, the better the chances that the simulation model will be devised to attain that. Toward that end, the executive must select the outlook and level of detail of simulation. This signifies the unavoidable standard of complicatedness of the model and the news necessities of the study.

***The second task is model development***. Typically, this includes determining on the form of the model and utilizing a computer to complete activity the simulations. (For teaching purposes, the instances and questions in this place episode are generally manual, but in most actual-life practically computers are used. This stems from the need for abundant numbers of runs, the complicatedness of simulations, and the need for the act of one that records of results.) Data accumulation is a important constituent model happening. The amount and type of facts wanted are a direct function of the sphere and level of detail of the simulation. The facts are required for both model development and judgment. Naturally, the model must be planned to authorize judgment of key resolution opportunities.

***The third step* that is *validation (Testing) phase*** is approximately had connection with model happening. Its main purpose search out decide if the model adequately describes evident method efficiency. An investigator usually achieves this by equating the results of simulation runs accompanying popular performance of bureaucracy under the unchanging chances. If aforementioned a contrasting cannot be made cause, for instance, actual-life data are difficult or impossible to acquire, an alternative search out engage a test of fairness, in which day of reckoning and belief of things used to bureaucracy or similar plans are depended for validation that the results are reasonable and agreeable. Still another aspect of confirmation is cautious concern of the arrogance of the model and the principles of parameters used in experiment the model. Again, day of reckoning and belief of those adept the real-life structure and those the one must use the results are essential. Finally, note that model development and model validation be similar or consistent: Model deficiencies exposed all along confirmation prompt model revisions, that bring about the need for further validation works and possibly further revisions.

***The fourth step in simulation is designing experiments***. Experiments are the character of a simulation; they help answer the *what-if* questions formal in simulation studies. By searching the process, the manager or predictor learns about creature nature.

***The fifth step is to run the simulation model***. If a simulation model is deterministic as well all parameters are known and constant, only a particular run will command a price of each what if question. But if the model is probabilistic, accompanying parameters liable to be subjected chance variability,manifold runs will be needed to get a clear image of the results. In this manual, probabilistic simulations are the focus of the consideration, and the comments are restricted to bureaucracy. Probabilistic simulation is basically a form of random examination, accompanying each run representing individual observation. Therefore, mathematical hypothesis maybe used to decide appropriate sample sizes. In effect, the best the strength of instability owned by simulation outcomes, the greater the number of simulation runs required to solve a justifiable level of assurance in the results as accurate signs of model nature.

***The last step in the simulation process is to analyze and interpret the results***. Explanation of the results depends to a big magnitude on the grade at which point the simulation model approximates realism; the nearer the estimate, the less need to "modify" the results. Moreover, the nearer the estimate of the model to real world, the less the risk intrinsic in employing the results.

## 4.5 TYPES OF SIMULATION MODELS TO CONTROL IN MANAGEMENT SCIENCE

There are in general four types of Simulation models that is to say as:

1. **Monte Carlo/ Risk Analysis Simulation**
2. **Agent Based Modeling and Simulation**
3. **Discrete Event Simulation**
4. **System Dynamics Simulation Solutions**

- **Monte Carlo/ Risk Analysis Simulation**

In simple words, a Monte Carlo simulation is a arrangement of risk evaluation. Businesses use it just before executing a main project or change in a process, in the way that a production manufacturing system.

Built on analytical models, Monte Carlo studies use the practical data of the authentic system's inputs and outputs (for example, supply consumption and manufacture yield). It before identifies doubts and potential risks through possibility distributions.

The benefit of a Monte Carlo-based simulation is that it determines knowledge and a all-encompassing understanding of potential warnings to your basic and period-to-market.

You can implement Monte Carlo simulations to almost some corporation or field, containing oil and gas, production, engineering, supply chain administration, and many possible choice.

We will survey the Monte Carlo Simulation in concisely in the next sections of this chapter.

- **Agent Based Modeling and Simulation**

An agent-based simulation is a model that examines the impact of an 'agent' on the 'system' or 'atmosphere.' In simple terms, just plan the impact a new laser-cutting tool or some other firm apparatus has on your overall manufacturing line.

The 'agent' in agent-based models maybe public, equipment, and practically whatever other. The simulation contains the agent's 'performance,' that be a part of rules of by means of what those agents must act in bureaucracy. You formerly examine by virtue of what bureaucracy responds to those rules.

However, you must draw your rules from real-globe data -alternatively, you will not create correct insights. In a habit, it serves by way of to analyze a projected change and identify potential risks and time.

- **Discrete Event Simulation**

A individual happening simulation model enables you to note the particular performances that influence your trade processes. For example, the typical mechanics support process includes the end-user calling you, your system taking and allocating the call, and your agent picking up agreement.

You would use a discrete happening simulation model to test that mechanics support process. You can use individual event simulation models to study many types of structures (for instance, healthcare, production, etc), and for a various range of outcomes.

For example, the Nebraska Medical Center had used individual happening simulation models to visualize how it manage discard productivity bottlenecks, increase the application of its functioning rooms, and lower sufferer/specialist travel distance and occasion.

- **System Dynamics Simulation Solutions**

This is a very abstract form of simulation modeling. Unlike agent-based modeling and individual case modeling, structure movement does not involve particular analyses about the system. So for a production ability, this model will not influence in data about the system and labor.

Somewhat, trades would use structure movement models to simulate for a long-term, strategic-level view of the overall structure.

In other words, the preference be going to grab aggregate-level observations about the entire structure in reaction to an operation — like, a decline in CAPEX, outcome a product line, etc.

## 4.6 MONTE CARLO SIMULATION

There are many various types of simulation methods. The consultation will devote effort to something probabilistic simulation utilizing the Monte Carlo procedure. The method gets its name from the famous Mediterranean resort lead games of chance. The chance factor is an main situation of Monte Carlo simulation, and this approach maybe used only when a process has a chance, or chance, component.

In the Monte Carlo arrangement, a administrator labels a frequency distribution that indicates the chance component of the system understudy. Random samples captured from this frequency distribution are similar to findings created on the system itself. As the number of findings increases, the results of the simulation will more carefully approximate the nature of the actual system, given an appropriate model has been developed. Sampling is consummate apiece use of chance numbers.

The elementary steps in the process are in this manner:

1. Describe a frequency distribution individually chance component of the system.
2. Figure out an assignment so that intervals of chance numbers will pertain the frequency distribution.
3. Acquire the chance numbers required for the learning.
4. Explain the results.

The chance numbers used in Monte Carlo simulation can arise some source that exhibits the essential randomness. Typically, they derive from one of two sources: Large studies believe computer-generated chance numbers, and limited studies usually use numbers from a table of random digits like the one demonstrated in Table 19S-1. The digits are listed in pairs for availability, but they maybe used individually, in pairs, or in whatsoever combination some given issue demands.

Two main appearance of the sets of chance numbers are owned by simulation. One is that process are evenly delivered. This way that for some height arrangement of digits (for example, two-digit numbers), each probable outcome (for instance, 34, 89, 00) has the same possibility of performing. The second feature is that there are no distinct shapes in sequences of numbers to implement individual to conclude numbers further in the series (so the name chance digits). This feature holds for some series of numbers; the numbers can be interpret across rows and up or down columns.

**TABLE 19S-1**

*Random digits*

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 18 | 20 | 84 | 29 | 91 | 73 | 64 | 33 | 15 | 67 | 54 | 07 |
| **2** | 25 | 19 | 05 | 64 | 26 | 41 | 20 | 09 | 88 | 40 | 73 | 34 |
| **3** | 73 | 57 | 80 | 35 | 04 | 52 | 81 | 48 | 57 | 61 | 29 | 35 |
| **4** | 12 | 48 | 37 | 09 | 17 | 63 | 94 | 08 | 28 | 78 | 51 | 23 |
| **5** | 54 | 92 | 27 | 61 | 58 | 39 | 25 | 16 | 10 | 46 | 87 | 17 |
| **6** | 96 | 40 | 65 | 75 | 16 | 49 | 03 | 82 | 38 | 33 | 51 | 20 |
| **7** | 23 | 55 | 93 | 83 | 02 | 19 | 67 | 89 | 80 | 44 | 99 | 72 |
| **8** | 31 | 96 | 81 | 65 | 60 | 93 | 75 | 64 | 26 | 90 | 18 | 59 |
| **9** | 45 | 49 | 70 | 10 | 13 | 79 | 32 | 17 | 98 | 63 | 30 | 05 |
| **10** | 01 | 78 | 32 | 17 | 24 | 54 | 52 | 44 | 28 | 50 | 27 | 68 |
| **11** | 41 | 62 | 57 | 31 | 90 | 18 | 24 | 15 | 43 | 85 | 31 | 97 |
| **12** | 22 | 07 | 38 | 72 | 69 | 66 | 14 | 85 | 36 | 71 | 41 | 58 |

When utilizing the table, it is main to prevent forever offset in the uniform spot; that would effect the same series of numbers each period. Various procedures endure for selecting a chance starting point. One can use the serial number of a model~r;bill to select the row, column, and way of number choice. Or use rolls of a die. For our purposes, the starting point will be described in each manual case or problem because all obtains the equivalent results.

The process of simulation will become more transparent as we solve some simple problems.

**Example-1:**

The supervisor of a manufacturing plant is worried about automobile breakdowns. He be able a conclusion to pretend breakdowns for a 10-day duration. Historical info on breakdowns over the last 100 days are likely in the following table:

| Number of Breakdowns | Frequency |
|---|---|
| 0………. | 10 |
| 1………. | 30 |
| 2………. | 25 |
| 3………. | 20 |
| 4………. | 10 |
| 5………. | 5 |
| | - |
| | 100 |

Simulate breakdowns for a 10-day duration. Read two-number random numbers from Table 19S-l, starting at the top of column 1 and study down.

a) Develop cumulative frequencies for breakdowns:

1. Convert frequencies into relative frequencies by dividing each frequency by the sum of the frequencies. Thus, 10 turns into 10/100 = .10, 30 turns into 30/100 =.30, and so forth.
2. Develop cumulative frequencies by successive summing. The results are demonstrated in the following table:

| **Number of Breakdowns** | **Frequency** | **Relative Frequency** | **Cumulative Frequency** |
|---|---|---|---|
| 0………. | 10 | .10 | .10 |
| 1………. | 30 | .30 | .40 |
| 2………. | 25 | .25 | .65 |
| 3………. | 20 | .20 | .85 |
| 4………. | 10 | .10 | .95 |
| 5………. | 5 | .05 | 1.00 |
| | - | - | |
| | 100 | 1.00 | |

a) Assign random-number breaks to relate the cumulative frequencies for breakdowns. (Note: Use two-digit numbers as the repetitions are likely to two decimal places.) You want a 10 percent chance of collecting the event "0 breakdowns" in our simulation. Thus, you must assign 10 percent of the probable random numbers as corresponding to that event. There are 100 two-digit numbers, so we can designate the 10 numbers 01 to 10 to that event.

Similarly, designate the numbers 11 to 40 to "one breakdown," 41 to 65 to "two breakdowns," 66 to 85 to "three breakdowns," 86 to 95 to "4 breakdowns" and 96 to 00 to "five breakdowns".

| Number of Breakdowns | Frequency | Relative Frequency | Cumulative Frequency | Corresponding Random Numbers |
|---|---|---|---|---|
| 0………. | 10 | .10 | .10 | 01 to 10 |
| 1………. | 30 | .30 | .40 | 11 to 40 |
| 2………. | 25 | .25 | .65 | 41 to 65 |
| 3………. | 20 | .20 | .85 | 66 to 85 |
| 4………. | 10 | .10 | .95 | 86 to 95 |
| 5………. | 5 | .05 | 1.00 | …………. |
| | - | - | | |
| | 100 | 1.00 | | |

a) Obtain the random numbers from Table 19S-1, column 1, as stated in the question: 18 25 73 12 54 96 23 31 45 01

b) Convert the random numbers into numbers of breakdowns:

18 falls in the intervening time 11 to 40 and corresponds, then, to one breakdown on day 1. 25 falls in the intervening time 11 to 40, this corresponds to one breakdown on day 2.

73 corresponds to three breakdowns on day 3.

12 corresponds to one breakdown on day 4.

54 corresponds to two breakdowns on day 5.

96 corresponds to five breakdowns on day 6.

23 corresponds to one breakdown on day 7.

31 corresponds to one breakdown on day 8.

45 corresponds to two breakdowns on day 9.

01 corresponds to no breakdowns on day 10.

The following table compiles these results:

| Days | Random Number | Simulated Number of Breakdowns |
|---|---|---|
| 1 | 18 | 1 |
| 2 | 25 | 1 |
| 3 | 73 | 3 |
| 4 | 12 | 1 |
| 5 | 54 | 2 |
| 6 | 96 | 5 |
| 7 | 23 | 1 |
| 8 | 31 | 1 |
| 9 | 45 | 2 |
| 10 | 01 | 0 |
| | | - |
| | | 17 |



The mean number of breakdowns for this 10-period simulation is 17110 = 1.7 breakdowns per day. Compare this to the predicted number of breakdowns based on the historical data:

$0(.10) + 1(.30) + 2(.25) + 3(.20) + 4(.10) + 5(.05) = 2.05$ per day

Following are the various points to be noticing:

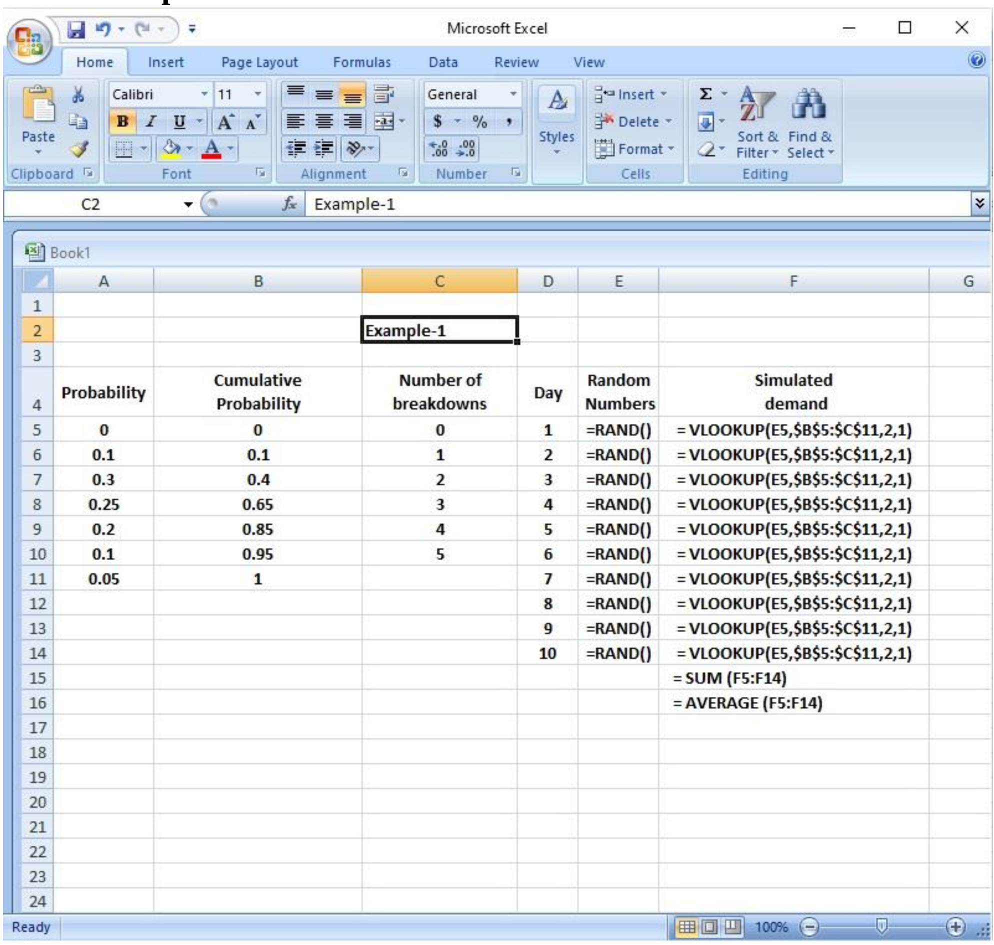
1. This simple model is proposed to represent the fundamental idea of Monte Carlo simulation. If our only aim search out estimate the average number of

breakdowns, we would not ought simulate; we commit base the estimate on the historical data only.

2. The simulation endure be considered as a sample; it is completely likely that extra runs of 10 numbers would produce different means
3. Because of the irregularity owned by the results of small samples, it hopeful foolish to attempt to draw any firm decisions from them; in an real study, much larger sample sizes almost used.

In few cases, it is beneficial to assemble a flowchart that defines a simulation, particularly if the simulation will include periodic updating of system values (for instance, amount of stock available), as illustrated in Example-2. The Excel computer program expression for this question is demonstrated below. Note that the adjustment of values in columns B, C, and E must be exactly as revealed.

**Code for Input:**



| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | Example-1 | | | |
| 3 | | | | | | |
| 4 | Probability | Cumulative Probability | Number of breakdowns | Day | Random Numbers | Simulated demand |
| 5 | 0 | 0 | 0 | 1 | =RAND() | = VLOOKUP(E5,$B$5:$C$11,2,1) |
| 6 | 0.1 | 0.1 | 1 | 2 | =RAND() | = VLOOKUP(E5,$B$5:$C$11,2,1) |
| 7 | 0.3 | 0.4 | 2 | 3 | =RAND() | = VLOOKUP(E5,$B$5:$C$11,2,1) |
| 8 | 0.25 | 0.65 | 3 | 4 | =RAND() | = VLOOKUP(E5,$B$5:$C$11,2,1) |
| 9 | 0.2 | 0.85 | 4 | 5 | =RAND() | = VLOOKUP(E5,$B$5:$C$11,2,1) |
| 10 | 0.1 | 0.95 | 5 | 6 | =RAND() | = VLOOKUP(E5,$B$5:$C$11,2,1) |
| 11 | 0.05 | 1 | | 7 | =RAND() | = VLOOKUP(E5,$B$5:$C$11,2,1) |
| 12 | | | | 8 | =RAND() | = VLOOKUP(E5,$B$5:$C$11,2,1) |
| 13 | | | | 9 | =RAND() | = VLOOKUP(E5,$B$5:$C$11,2,1) |
| 14 | | | | 10 | =RAND() | = VLOOKUP(E5,$B$5:$C$11,2,1) |
| 15 | | | | | | = SUM (F5:F14) |
| 16 | | | | | | = AVERAGE (F5:F14) |

The simulation results are demonstrated in the following screen. Use key F4 do to a simulation or additional simulation.

**Output:**

Example-1

| Probability | Cumulative Probability | Number of breakdowns | Day | Random Numbers | Simulated demand |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0.18475 | 1 |
| 0.1 | 0.1 | 1 | 2 | 0.94169 | 4 |
| 0.3 | 0.4 | 2 | 3 | 0.40460 | 2 |
| 0.25 | 0.65 | 3 | 4 | 0.69092 | 3 |
| 0.2 | 0.85 | 4 | 5 | 0.80310 | 3 |
| 0.1 | 0.95 | 5 | 6 | 0.06175 | 0 |
| 0.05 | 1 | | 7 | 0.67353 | 3 |
| | | | 8 | 0.15092 | 1 |
| | | | 9 | 0.20887 | 1 |
| | | | 10 | 0.69478 | 3 |
| | | | | | 21 |
| | | | | | 2.1 |

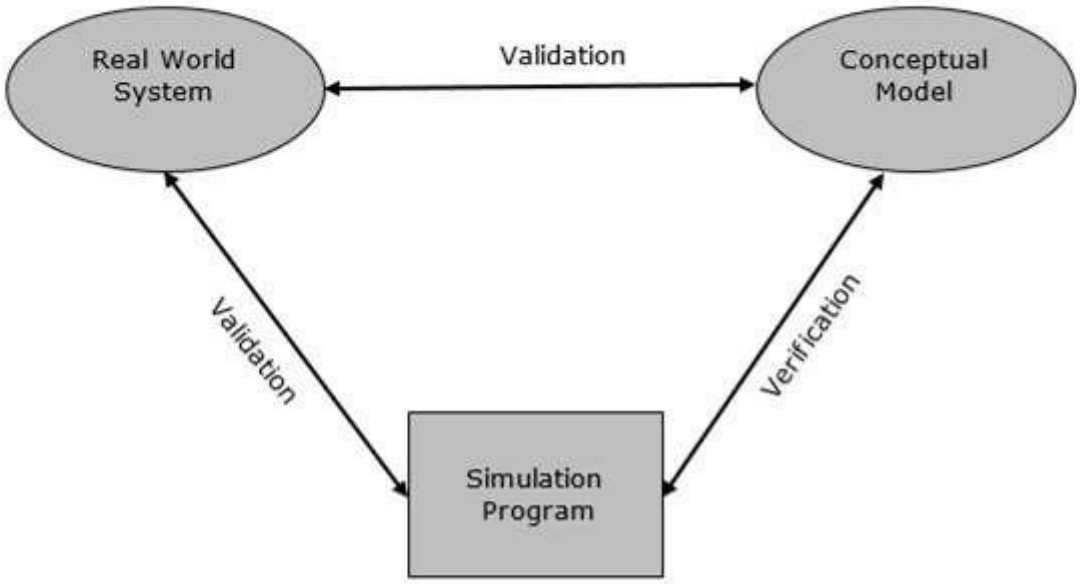
## 4.7 TOOLS FOR THE VERIFICATION AND VALIDATION OF SIMULATION MODEL

One of the authentic questions that the simulation investigator faces is to validate the model. The simulation model is accurate only if the model is an correct description of the original structure, else it is invalid.

Validation and verification are the two steps in any simulation project to validate a model.

- **Validation** is the method of matching two results. In this method, we require to match the description of a theoretical model to the real structure. If the comparison is correct, therefore it is valid, otherwise invalid.

- **Verification** is the method of equating two or more results to insure its accuracy. In this method, we should equate the model's application and its combined data among the developer's visionary report and specifications.




There are various techniques used to perform Verification & Validation of Simulation Model. Following are some of the common techniques –

There are several methods used to act Verification & Validation of Simulation Model. Following are few of the ordinary methods –

4.7.1. METHODS TO EXECUTE VERIFICATION OF SIMULATION MODEL

Following are the approaches to act verification of simulation model-

✓ Through utilizing programming techniques to address and debug the program in substitute-programs.

- ✓ Through utilizing “Structured Walk-through” procedure in which more than one person be going to explain the program.
- ✓ Through following the intermediate results and matching them with noticed results.
- ✓ Through testing the simulation model output utilizing various input combos.
- ✓ Through matching last simulation result with investigative results.

### 4.7.2. METHODS TO EXECUTE VALIDATION OF SIMULATION MODEL

**Step 1** − Create a model accompanying extreme validity. This maybe made employing the following steps −

- ✓ The model must be conferred accompanying management experts while crafty.
- ✓ The model must communicate accompanying the customer during the whole of the process.
- ✓ The output must directed by method specialists.

**Step 2** − Test the model at hypotheses data. This maybe attained by referring the hypothesis data into the model and examination it quantitatively. Sensitive study can further be acted to see the effect of change in the consequence when important changes are created in the input data.

**Step 3** − Determine the representative productivity of the Simulation model. This maybe obtained using the following steps −

- ✓ Determine by means of what close is the simulation output accompanying the real structure output.
- ✓ Comparison maybe acted using the Turing Test. It presents the data in management pattern, that maybe explained by specialists only.
- ✓ Statistical plan maybe used for equate the model output accompanying the actual system output.

### 4.7.3. MODEL DATA COMPARISON WITH REAL FACTS

After created the model, we have to display comparison of its output data with original system data. Following are the two ways to execute this contrasting.

#### 4.7.3.1. VALIDATING THE CURRENT SYSTEM

In this technique, we use real-realm inputs of the model to equate its output with that of the real-world inputs of the original structure. This process of validation is straight forward, still, it can give few problems when achieved, in the way that if the output be going to be compared to average distance, awaiting time, useless period, etc. it maybe distinguished using statistical tests and hypothesis test. Some of the statistical tests are Chi-square test, Kolmogorov-Smirnov test, Cramer-von Mises test, and the Moments test.

#### 4.7.3.2. VALIDATING THE FIRST TIME MODEL

Consider we should describe a expected structure which doesn't lie at the nor has existed earlier. Therefore, skilled is no historical data feasible to compare its efficiency with. Hence, we should use a supposed system formed on assumptions. Following useful hints will help in making it effective.

- **Subsystem Validity** – A model itself can not have any existing structure to match it accompanying, but it may comprise a famous subsystem. Each of that validity can be tested alone.
- **Internal Validity** – A model accompanying large size of internal fluctuation will be rejected as a stochastic system accompanying extreme variance on account of its in-house processes will hide the changes in the output due to input changes.
- **Sensitivity Analysis** – It specifies the facts about the sensitive parameter in management at which point we need to pay larger attention.
- **Face Validity** – When the model acts on opposite logics, before it bear be rejected although it behaves like the actual structure.

## 4.9 ADVANTAGES AND LIMIATIONS OF SIMULATION

Among the main benefits of simulation are these:

- It lends itself to questions that are difficult or preposterous to resolve mathematically.
- It permits an investigator to experiment accompanying structure performance while avoiding probable risks owned by testing accompanying the original structure.
- It compresses time for fear that managers can fast ascertain complete belongings.
- It can be a part of a valuable tool for training conclusion creators by construction up their occurrence and understanding of structure nature under a broad range of situations.
- In Logistics: In the ever shorter implementation periods of logistics projects, simulation is a tool that makes a lasting contribution to planning reliability and thus to the success of a project. The key factor in the successful use of simulation is fast and qualified modeling.
- In Production: Simulation safeguards your investments in machines and means of production and differs from classical investment calculation, which is designed for local optimization.
- It is a useful technique for solving a business problem where many values of the variables are not known or partly known in advance and there is no easy way to find these values.

Further, Application areas for simulation are practically unlimited. Today simulation can be used for decision-support with supply chain management, workflow and throughput analysis, facility layout design, resource usage and allocation, resource management and process change. Whether contemplating a new office building, planning a new factory design, assessing predictive and reliability maintenance, anticipating new or radical procedures, deploying new staff, or planning a day's activities, simulation can play a crucial role in finding the right and timely solutions. The progressive and technology driven organizations, in pursuit of winning and/or maintaining their market share, have taken different approaches to their success. In their pursuit, some have focused on "customer service", many have embraced the "productivity" theme, and yet others have pursued the important issue of "quality and reliability". In recent times, simulation has been very successfully used as a modeling and analysis tool.

However, certain conditions are too accompanying simulation. Leading with these are:

1. Simulation does not produce an best result; it quite signifies an similar nature for a given batch of inputs.
    a) By strategy, there is fundamental randomness (that is, chance numbers) in simulation.
    b) Simulations are formed on models, and models are simply estimates of actuality.

2. For extensive simulation, it can compel important endeavor to establish a suitable model as well substantial computer time to access simulations.

Because simulation generates an approximate answer in place of an exact answer, and because of the cost of running a simulation study, simulation is not regularly the first choice of a conclusion creator. Alternatively, contingent upon the complexity of the condition, instinctive or investigative designs should first be examined. In simple cases, an instinctive solution generally is sufficient. In more complicated cases, an analytical resolution is desirable, supposing an appropriate technique is convenient. If not, possibly probable to develop an systematic model that maybe used to achieve a result. If these estimates do not enough, simulation turns into the next probable possibility. By all means, if particularly not economically admissible, the conclusion creator will have to depend on decision and knowledge; in actuality, later reevaluating all of the possible choices, the decision creator can go back to an instinctive solution, however at the beginning that approach performed not clear adequate.

## 4.10 SUMMARY

- In this Chapter, at first, we investigate establishment of Simulations
- The next point in discussion has been on the various kinds of Simulations.
- The next stage in consideration has done on the *Various Kinds of Simulations.*
- A Monte-Carlo Study has abided introduced accompanying numerical examples

through Excel Spreadsheet computation.

- The Validation and Verification methods have been studied.

## 4.10 SELF-ASSESSMENT QUESTIONS

1. What do you mean by word "*Simulation*"?
2. What are some of the primary reasons for the widespread use of simulation techniques in practice?
3. What are few of the basic reasons for the extensive use of Simulation methods in essence?
4. What are few of the manners managers can use Simulation?
5. What act do random numbers perform in Monte Carlo simulations?
6. List the key benefits of Simulation.
7. Interpret the Verification and Validation processes.



## 4.11 OBJECTIVE QUESTIONS

1. The process of Simulation
   (a) is acknowledged as "Monte-Carlo" Simulation.
   (b) is a strong analytical technique.
   (c) consistently demand use of computers to calculate solutions for the problems.
   (d) include the test in what way the result of Simulation model is autonomous of the simulation run.

2. Simulation in the framework of trade problems
   (a) does not produce best results.
   (b) are relatively more sensible than mathematical forms.
   (c) relatively more high-priced system of analysis.
   (d) all the above

3. Simulation is
   (a) valuable to analyse problems where examining result is different.
   (b) a statistical experiment as such its outcomes are based on statistical errors.
   (c) definitive in type.
   (d) all the above

4. Simulation concede possibility not be applied entirely cases as it
   (a) pay much computer time.
   (b) needs substantial expertise for model construction and huge computer programming exercises.
   (c) provides somewhat approximate result to problem.
   (d) all the above.

5. Large complex simulation models are comprehended, cause
   (a) they can be high-priced to write and use as an empirical instrument.
   (b) their average costs are not clear.
   (c) it is complicated to build the relevant happenings.
   (d) all the above.

6. Analytical results are captured into concern before a simulation study so as to
   (a) decide the best result.
   (b) determine acceptable values of decision variables for the particular selections of system parameters.
   (c) recognize suitable values of the system parameters.
   (d) all the above.

7. In Monte- Carlo Simulation
   (a) the key necessity is randomness.
   (b) the model is of deterministic type.
   (c) random numbers maybe used to produce the value of input variables particularly, if the sampled distribution is uniform.
   (d) none of the above.

8. As simulation is not an Analytical model, accordingly, result of simulation must be considered as
   (a) impractical

(b) accurate
(c) estimate
(d) simplified.

9. Key benefits of Simulation for trade contains
   (a) Flexbility
   (b) Time condensation
   (c) Test Large and/or Complex Systems
   (d) all the above

10. Which of the following charge is not correct?
    (a) Elementary footsteps in the use of simulation method are approximately free of the nature of the problem.
    (b) Simulation includes cultivating a model of few actual phenomenon and then analyzing on it.
    (c) Simulation can not be used when analytical tools maybe used.
    (d) Probabilistic simulation is like random sampling where output is based on statistical error.

11. One can boost the probability that results of simulation are not invalid by
    (a) using individual probability distribution in place of continuous one.
    (b) validating the simulation model.
    (c) changing the input parameters.
    (d) none of the above.

12. Biased random sampling is made from among possible choices which have
    (a) different probability
    (b) equal possibility
    (c) possibility which do not sum to unity
    (d) none of the above.

13. Verification is the process
    (a) of different possibility
    (b) of matching two or more results to insure its preciseness
    (c) of changing the input parameters.
    (d) none of the above.

14. We can implement Monte Carlo simulations to basically some manufacturing firms or field, containing
    (a) oil and gas, production
    (b) engineering
    (c) supply chain administration
    (d) all of the above.

15. The chance component is an important feature of
    (a) Monte-Carlo Simulation
    (b) Agent Based Modeling and Simulation
    (c) System Dynamics Simulation Solutions
    (d) Discrete Event Simulation

**ANSWERS OF THE OBJECTIVE QUESTIONS**

**1.** (a) **2.** (d) **3.** (d) **4.** (d) **5.** (a) **6.** (b) **7.** (a) **8.** (c) **9.** (d) **10.** (c)
**11.** (b) **12.** (a) **13.** (b) **14.** (d) **15.** (a)



## 4.12 REFERENCES & SUGGESTED BOOKS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# LESSON 5
# DECISION MAKING UNDER UNCERTAINTY

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*"A decision is the conclusion of a process by which one choices between two or more available courses of action for the purpose of attaining a goal"*

## STRUCTURE

5.1 Learning Objectives
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5.3 Decision Making under uncertainty
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## 5.1 LEARNING OBJECTIVES

After completing this chapter, you will be able to:

1. List the steps of the decision-making process and describe the different types of decision-making environments.
2. Make decisions under uncertainty.
3. Make decisions under risk.
4. Develop accurate and useful decision trees

## 5.2 INTRODUCTION

Humans make a lot of decisions every day, and occasionally we make ones that could have a significant impact on our lives both now and in the future. The capability of making good judgments on time has a significant impact on the success or failure that an individual or organisation experiences. We would prefer to make the right choice when it comes to key decisions like where to attend college, whether to buy or rent a car, and other similar choices.

When a decision maker is presented with multiple option possibilities and an unclear or risk-filled pattern of future occurrences, decision analysis can be utilised to create the best course of action. In order to decide whether to deploy a medical screening test to identify metabolic problems in neonates, for instance, The State of North Carolina conducted decision analysis. Decision analysis therefore consistently demonstrates its value in decision making. Even when a thorough decision analysis has been performed, unforeseen future circumstances cast doubt on the outcome. The chosen decision alternative may occasionally produce good or great results. In other circumstances, a hypothetical future occurrence might materialise and render the chosen decision alternative only mediocre or worse. The uncertainty surrounding the outcome is a direct cause of the risk attached to any chosen alternative. Risk analysis is a key component of a sound decision analysis. The decision-maker is given probability information about both potential positive and negative outcomes through risk analysis.

Decision making under risk and uncertainty is a fact of life. In decision making under pure uncertainty, the decision maker has no knowledge regarding any of the states of nature outcomes, and/or it is costly to obtain the needed information. There are many ways of handling unknowns when making a decision. We will try to enumerate the most common methods used to get information prior to decision making under risk and uncertainty.

## 5.3 DECISION MAKING UNDER UNCERTAINTY

When making decisions under uncertainty, decision makers are completely in the dark regarding the likelihood of various outcomes. In other words, they are unsure of how likely (or unlikely) a particular scenario is. For instance, it is impossible to forecast the likelihood that Mr. X will serve as the nation's prime minister for the ensuing 15 years.

When it is impossible to quantify the probability of a result, the decision-maker must base their choice only on the conditional payoff values themselves, keeping the effectiveness standard in mind.

### 5.3.1 *Decision Criteria*

Under conditions of uncertainity, only payoffs are known and the chance of occurrence any state of nature is unknown. The following are the criteria of decision making under uncertainity:

(i) Optimism (Maximax or Minimin) criterion
(ii) Pessimism (Maximin or Minimax) criterion
(iii) Equal Probabilities (Laplace) criterion
(iv) Coefficient of optimism (Hurwiez) criterion
(v) Regret (Salvage) criterion

#### 5.3.1.1 Optimism (Maximax or Minimin) Criterion

This criterion ensures that the decision-maker doesn't miss the chance to choose the particular strategy that correspond to largest possible profit (maximax) or the lowest possible cost (minimin). So, out of all the alternatives, he selects the decision alternative that maximizes the maximum payoff (or minimizes the minimum payoff).

The following is the working method of this criterion:

Step 1: Determine the maximum (or minimum) possible payoff corresponding to each alternative.

Step 2: Select that decision alternative which corresponds to the maximum (or minimum) of the above maximum (minimum) payoffs.

Because this criterion finds the option with the overall highest reward feasible while adopting a very optimistic future outlook, it is called the *optimistic* criterion.

**Example 1:** A food products company is contemplating the introduction of a revolutionary new product with new packaing or replacing the existing product at much higher price $(S_1)$. It may even make a moderate change in the composition of the existing product, with a new packaging at a small increase in price $(S_2)$, or may mall a small change in the composition of the existing product, backing it with the word 'New' and a negligible in crease in price $(S_3)$. The three possible states of nature or events are: (i) high increase in sales $(N_1)$, (ii) no change in sales $(N_2)$ and (iii) decrese in sales $(N_3)$. The marketing department of the company worked out the payoffs in terms of yearly net profits for each of the strategies of three events (expected sales). This is represented in the following table:

**Table 1**

| *Strategies* | *States of Nature* $N_1$ | $N_2$ | $N_3$ |
|---|---|---|---|
| $S_1$ | 7,00,000 | 3,00,000 | 1,50,000 |
| $S_2$ | 5,00,000 | 4,50,000 | 0 |
| $S_3$ | 3,00,000 | 3,00,000 | 3,00,000 |

Which strategy should the concerned executive choose using *optimistic* criterion?

**Solution:** In Table 2 we see that using optimistic criterion executive's maximax choice is the first Strategies, $(S_1)$. The 7,00,000 payoff is the maximum of the maximum payoffs (i.e., 7,00,000, 5,00,000, and 3,00,000) for each Strategies.

**Table 2**

| *States of Nature* | *Strategies* $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|
| $N_1$ | 7,00,000 | 5,00,000 | 3,00,000 |
| $N_2$ | 3,00,000 | 4,50,000 | 3,00,000 |
| $N_3$ | 1,50,000 | 0 | 3,00,000 |
| *Column (maximum)* | **7,00,000** | 5,00,000 | 3,00,000 |
| | ↑ Maximax Payoff | | |

#### 5.3.1.2 Pessimism (Maximin or Minimax) criterion

This criterion is based on the "conservative approach," which holds that the worst-case scenario will occur. For this reason, it is called the *pessimistic* criterion. Decision-makers choose those alternative that, in the case of gains, correspond to the maximum of the minima values (or, in the event of a loss, the minimum of the maxima values).

The working method of this criterion is as follows:

Step 1: Determine the minimum (or maximum) possible cost for each alternative.

Step 2: Choose that alternative which corresponds to the maximum of the above minimum payoffs (or minimum of the above maximum cost).

**Example 2:** Use the data given in example 1 and find that which strategy should the concerned executive choose using *pessimistic* criterion?

**Solution:** In Table 3 we see that using pessimistic criterion executive's maximin choice is the first Strategies, $(S_3)$. The 3,00,000 payoff is the maximum of the minimum payoffs (i.e., 1,50,000, 0, and 3,00,000) for each Strategies.

**Table 3**

| *States of Nature* | *Strategies* | | |
|---|---|---|---|
| | $S_1$ | $S_2$ | $S_3$ |
| $N_1$ | 7,00,000 | 5,00,000 | 3,00,000 |
| $N_2$ | 3,00,000 | 4,50,000 | 3,00,000 |
| $N_3$ | 1,50,000 | 0 | 3,00,000 |
| *Column (minimum)* | 1,50,000 | 0 | 3,00,000 |
| | | | ↑ Maximin Payoff |

#### 5.3.1.3 Equal Probabilities (Laplace) criterion

Since the probabilities of states of nature are unknown, it is assumed that all states of nature will occur with equal probability meaning that all possible events have an equal chance of happening.

The working method are as follows:

Step 1: Assign equal probability value to each state of nature by using the formula:

1÷ (*number of the states of nature*)

Step 2: Compute the expected (or average) payoff for each alternative (course of action) by adding all the payoffs and dividing by the number of possible states of nature, or by applying the formula:

(*Probability of state of nature j*)× (*Payoff value for the combination of alternative I and state of nature j*.)

Step 3: Select the best expected payoff value (maximum for profit and minimum for cost).

**Example 3:** Use the data given in example 1 and find that Which strategy should the concerned executive choose using *equal probabilities* criterion?

**Solution:** Assuming that each state of nature has a probability 1/3 of occurrence. Thus, from table 4, using equal probabilities criterion, we see that the largest expected return is from strategy $(S_1)$, the executive must select strategy $(S_1)$.

**Table 4**

| *Strategies* | *States of Nature* | | | Expected Return (Rs.) | | |
|---|---|---|---|---|---|---|
| | $N_1$ | $N_2$ | $N_3$ | | | |
| $S_1$ | 7,00,000 | 3,00,000 | 1,50,000 | (7,00,000+3,00,000+1,50,000)/3=3,83,333.33 | ← | Largest Payoff |
| $S_2$ | 5,00,000 | 4,50,000 | 0 | (5,00,000+4,50,000+0)/3=3,16,666.66 | | |
| $S_3$ | 3,00,000 | 3,00,000 | 3,00,000 | (3,00,000+3,00,000+3,00,000)/3=3,00,000 | | |

**5.3.1.4 Coefficient of optimism (Hurwiez) criterion**

According to this criterion, the decision-makers rarely exhibit excessive pessimism or optimism. The Hurwicz decision criterion (or criterion of optimism) offers a balance between optimistic and pessimistic decisions because most people tend to fall somewhere in the middle of the two extremes. By balancing them with varying degrees of optimism and pessimism, this gives a mechanism for striking a balance between extremes of both optimism and pessimism.

The working method of this criterion is given below:

Step 1: Chosse an appropriate degree of optimism (or pessimism) of the decision maker. Let $\alpha$ be his degree of optimism and then $(1-\alpha)$ be the degree of pessimism. $[0 \leq \alpha \leq 1]$.

Step 2: Determine the maximum as well as minmum payoff for each alternative and obtain the quantities

$$h = \alpha \times maximum + (1 - \alpha) \times minimum for\ each\ alternative.$$

**Example 4:** A manufacturer manufactures a product, of which the principal ingredient is a chemical X. At the moment, the manufacturer spends Rs 1,000 per year on supply of X, but there is a possibility that the price may soon increase to four times its present figure because of a worldwide shortage of the chemical. There is another chemical Y, which the manufacturer could use in conjunction with athird chemical Z, in order to give the same effect as chemical X. chemical Y and Z would together cost the manufacturer Rs 3,000 per year, but their prices are unlikely to rise. If the coefficient of optimism is 0.4, then find the course of action that minimizes the cost?

**Solution:** The data of the problem is summarized in the following table (negative figures in the table shows profit).

**Table 5**

| States of Nature | Courses of Action | |
|---|---|---|
| | $S_1$ (use Y and Z) | $S_2$ (use X) |
| $N_1$ (Price of X increases) | -3,000 | -4,000 |
| $N_2$ (Price of X does not increase) | -3,000 | -1,000 |

Given the coefficient of optimism equal to 0.4, the coefficient of pessimism will be 1-0.4=0.6. Then according to Hurwicz, select course of action that optimizes (maximum for profit and minimum for cost) the payoff value

$$h = \alpha \times (Best\ payoff) + (1 - \alpha) \times (worst\ payoff)$$

$$= \alpha \times (maximum\ in\ column) + (1 - \alpha) \times (minimum\ in\ column)$$

**Table 6**

| *Course of Action* | *Best Payoff* | *Worst Payoff* | *h* |
|---|---|---|---|
| $S_1$ | -3,000 | -3,000 | -3,000 |
| $S_2$ | -1,000 | -4,000 | -2,800 |

Since course of action $S_2$ has the least cost (maximum profit) = 0.4(1,000) + 0.6(4,000) = Rs 2,800, the manufacturer should adopt this.

#### 5.3.1.5 Regret (Salvage) criterion

The final decision criterion that we explore is based on opportunity loss. This criterion is also called *opportunity loss decision criterion* or *minimax regret decision criterion*. The discrepancy between the optimal payoff and the actual payoff obtained is referred to as opportunity loss. In other words, it represents the amount lost as a result of choosing the wrong alternative. Regret (savage) identifies the decision inside each alternative that minimize the maximum opportunity loss.

The following is the algorithm for this criterion:

Step 1: From the given payoff matrix, develop an opportunity-loss (or regret) matrix as follows:

(i) Find the best payoff corresponding to each state of nature.
(ii) Subtract all other payoff values in that row from this value.

Step 2: For each decision alternative identify the worst (or maximum regret) payoff value. Record this value in the new row.

Step 3: Select a decision alternative resulting in a smallest anticipated opportunity loss value.

**Example 5:** Considering the same data of example 1 and find that Which strategy should the concerned executive choose using *regret* criterion?

**Solution:** The regret (opportunity-loss) table is shown below:

**Table 7**

| | *Strategies* | | | |
|---|---|---|---|---|
| *State of Nature* | $S_1$ | $S_2$ | $S_3$ | *Best Payoff* |
| $N_1$ | 7,00,000-7,00,000 = 0 | 7,00,000-5,00,000 = 2,00,000 | 7,00,000-3,00,000 = 4,00,000 | 7,00,000 |
| $N_2$ | 4,50,000-3,00,000 = 1,50,000 | 4,50,000-5,50,000 = 0 | 4,50,000-3,00,000 = 1,50,000 | 4,50,000 |
| $N_3$ | 3,00,000-1,50,000 = 1,50,000 | 3,00,000-0 = 3,00,000 | 3,00,000-3,00,000 = 0 | 3,00,000 |
| *Column (maximum)* | 1,50,000 | 3,00,000 | 4,00,000 | |
| | ↑ Minimax Regret | | | |

Hence the executive should adopt minimum opportunity loss strategy, $S_1$.

**IN-TEXT QUESTIONS**

9. A course of action that may be chosen by a decision maker is called an ……
10. In hurwicz describes making the coefficient of realism describes the degree of …….
11. …….. decision making criterion uses an opportunity loss decision.

## 5.4 RISK PROFILE OR DECISION UNDER RISK

The decision-maker has access to enough data to estimate the likelihood of each event (state of nature). A decision maker is considered to make risky decisions when he selects one alternative out of numerous that have known probabilities of occurrence. From the past data, several outcomes' probability can be calculated. The decision-maker may frequently base their choices on personal beliefs about what will happen in the future or on information gleaned from market research, the opinions of experts, etc. The issue can be resolved as a decision problem under risk.

Under the condition of risk, one of the most common ways of making decisions under risk is evaluating the alternative with the highest expected monetary value of the expected payoff. The ideas of expected opportunity loss and expected value of perfect information are also discussed.

### 5.4.1 Expected Monetary Value (EMV)

The expected monetary value (EMV) for a certain course of action is obatained be adding payoff values multiplied by the probabilities associated with each state of nature. Mathematically, EMV is stated as follows:

$$EMV\ (course\ of\ action, S_j) = \sum_{i=1}^{m} p_{ij} p_i$$

where, $m$ = number of possible states of nature

$p_i$ = probability of occurrence of state of nature, $N_i$

$p_{ij}$ = payoff associated with state of nature $N_i$ and course of action, $S_j$

The EMV criterion may be summarized as below:
Step 1: List conditional profit for each act-event combinations, along with the corresponding event probabilities.
Step 2: For each act, determine the expected conditional profits.
Step 3: Determine EMV for each act.
Step 4: Choose the act which corresponds to the optimal EMV.

**Example 6:** Mr X flies quite often from town A to town B. He can use the airport bus which costs Rs 25 but if he takes it, there is a 0.08 chance that he will miss the flight. The stay in a hotel costs Rs 270 with a 0.96 chance of being on time for the flight. For Rs 350 he can use a taxi which will make 99% chance of being on time for the flight. If Mr X catches the plane on time, he will conclude a business transaction that will produce a profit of Rs 10,000, otherwise he will lose it. Which mode of transport should Mr X use? Answer on the basis of the EMV criterion.

**Solution:** Computation of EMV associated with various courses of action is shown in table 8.

**Table 8**

| States of Nature | Courses of Action | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Bus | | | Stay in Hotel | | | Taxi | | |
| | Cost | Probability | Expected value | Cost | Probability | Expected value | Cost | Probability | Expected value |
| Catches the flight | 10,000-25 = 9,975 | 0.92 | 9,177 | 10,000-270 = 9,730 | 0.96 | 9,340.80 | 10,000-350 = 9,650 | 0.99 | 9,553.50 |
| Miss the flight | -25 | 0.08 | -2 | -270 | 0.04 | -10.8 | -350 | 0.01 | -3.5 |
| Expected monetary value (EMV) | | | 9,175 | | | 9,330 | | | 9,550 |

Since EMV associated with course of action 'Taxi' is largest (= Rs 9,550), it is the logical alternative.

### 5.4.2 Expected Opportunity Loss (EOL)

An alternative approach in decision making under risk is to minimize expected opportunity loss (EOL). Expected opportunity loss (EOL), also called *expected value of regret*. Mathematically, EOL is stated as follows:

$$EOL\ (State\ of\ Nature, N_i) = \sum_{i=1}^{m} l_{ij} p_i$$

where, $l_{ij}$ = opportunity loss due to state of nature, $N_i$ and course of action, $S_j$

$p_i$ = probability of occurrence of state of nature, $N_i$

Major steps in the EOL criterion may be summarized as below:

Step 1: List the conditional profit table for each act-event combination, along with corresponding event probabilities.

Step 2: For each event, determine the COL ( conditional opportunity loss) values by first locating the most favourable act (maximum payoff) for that event and ten taking the difference between that conditional profit value and each conditional profit for that event.

Step 3: For each act, determine the expected COL values and sum these values to get the expected opportunity loss (EOL) for that act.

Step 4: Choose that act which corresponds to the minimum COL values.

**Example 7:** A company manufactures goods for a market ggods in which the technology of the product is changing rapidly. The research and development department has produced a new product that appears to have potential for commercial exploitation. A further Rs 60,000 is required for development testing. The company has 100 customers and each coustomer might purchase, at the most, one unit of the product. Market research suggests that a selling price of Rs 6,000 for each unit, with the total variable costs of manufacturing and selling estimate as Rs 2,000 for each unit.

Form previous experience, it has been possible to derive a probability distribution relating to the proportion of customers who will buy the product as follows:

| | | | | | |
|---|---|---|---|---|---|
| Proportion of customers: | 0.04 | 0.08 | 0.12 | 0.16 | 0.20 |
| Probability: | 0.10 | 0.10 | 0.20 | 0.40 | 0.20 |

Determine the expected opportunity losses, given no other information than that stated above, and check whether or not the company should develop the product.

**Solution:** If p is the proportion of customers who purchase the new product, the company's conditional profit is: $(6{,}000 - 2{,}000) \times 100p - 60{,}000 = Rs(4{,}00{,}000p - 60{,}000)$.

Let $N_i (i = 1,2, \ldots ,5)$ be the possible states of nature, i.e. proportion of the customers who will buy the new product and $S_1$ (develop the product) and $S_2$ (do not develop the product) be the two courses of action.

The conditional profit values (payoffs) for each pair of $N_i's$ and $S_j's$ are shown in the table 9.

**Table 9**

| *Proportion of Customers (States of Nature)* | *Conditional Profit* = Rs (4,00,000p - 60,000) | |
|---|---|---|
| | $S_1$ (Develop) | $S_2$ (Do not Develop) |
| 0.04 | -44,000 | 0 |
| 0.08 | -28,000 | 0 |
| 0.12 | -12,000 | 0 |
| 0.16 | 4,000 | 0 |
| 0.20 | 20,000 | 0 |

Opportunity loss values are given below in table 10.

**Table 10**

| *Proportion of Customers (States of Nature)* | *Probability* | *Conditional Profit* (Rs) | | *Opportunity Loss* (Rs) | |
|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_1$ | $S_2$ |
| 0.04 | 0.1 | -44,000 | 0 | 44,000 | 0 |
| 0.08 | 0.1 | -28,000 | 0 | 28,000 | 0 |
| 0.12 | 0.2 | -12,000 | 0 | 12,000 | 0 |
| 0.16 | 0.4 | 4,000 | 0 | 0 | 4,000 |
| 0.20 | 0.20 | 20,000 | 0 | 0 | 20,000 |

Using the given estimates of probabilities associated with each state of nature, the expected opportunity loss (EOL) for each course of action is given below:

$$EOL(S_1) = 0.1(44{,}000) + 0.1(28{,}000) + 0.2(12{,}000) + 0.4(0) + 0.2(0) = Rs\ 9{,}600$$

$$EOL(S_2) = 0.1(0) + 0.1(0) + 0.2(0) + 0.4(4{,}000) + 0.2(20{,}000) = Rs\ 5{,}600$$

Since the company seeks to minimize the expected opportunity loss, the company should select course of action $S_2$ (do not develop the product) with minimum EOL.

### 5.4.3 Expected value of Perfect Information (EVPI)

Choosing a course of action that produces the intended results in the presence of any state of nature is simple if the decision maker is able to obtain flawless (complete and accurate) knowledge about the occurrence of various states of nature. The Expected value of perfect information (EVPI) may be defined as the maximum sum a person would be willing to pay to obtain perfect knowledge of which event would occur. Without any more information, the EMV or EOL criterion assists the decision-maker in choosing a specific course of action that maximises the expected payoff. Mathematically, it is stated as:

$EVPI = (expected\ profit\ with\ perfect\ information) - (expected\ profit\ without\ perfect\ information)$

$$= \sum_{i=1}^{m} p_i \max_j(p_{ij}) - EMV^*$$

where, $p_i$ = probability of occurrence of state of nature, $N_i$

$p_{ij}$ = best payoff when course of action, $S_j$ is taken in the presence of state of nature $N_i$

$EMV^*$ = maximum expected monetary value.

**Example 8:** XYZ company manufactures parts for passenger cars and sells them in lots of 10,000 parts each. The company has a policy of inspecting each lot before it is actually shipped to the retailer. Five inspection categories, established for quality control, represent the percentage of defective items contained in each lot. These are given in the following table. The daily inspection chart for past 100 inspections shown the following rating or breakdown inspection: Due to this the management in considering two possible course of action:

(i) $S_1$: Shut down the entire plant operations and thoroughly inspect each machine.

| *Rating* | *Proportion of Defective Items* | *Frequency* |
|---|---|---|
| Excellent (A) | 0.02 | 25 |
| Good (B) | 0.05 | 30 |
| Acceptable (C) | 0.10 | 20 |
| Fair (D) | 0.15 | 20 |
| Poor (E) | 0.20 | 5 |
| | Total | 100 |

(ii) $S_2$: Continue production as it now exists but offer the customer a refund for defective items that are discovered and subsequently returned.

The first alternative will costs Rs 600 while the second alternative will cost the company Rs 1 for each defective item that is returned. What is the optimum decision for the company? Find the EVPI.

**Solution:** Calculations of inspectiona and refund cost are shown in table 11.

**Table 11**

| Rating | Defective Rate | Probability | Cost | | Opportunity Loss | |
|---|---|---|---|---|---|---|
| | | | Inspect | Refund | Inspect | Refund |
| A | 0.02 | 0.25 | 600 | 200 | 400 | 0 |
| B | 0.05 | 0.30 | 600 | 500 | 100 | 0 |
| C | 0.10 | 0.20 | 600 | 1,000 | 0 | 400 |
| D | 0.15 | 0.20 | 600 | 1,500 | 0 | 900 |
| E | 0.20 | 0.05 | 600 | 2,000 | 0 | 1,400 |
| | | 1.00 | 600 | 670 | EOL=170 | 240 |

The cost of refund is calculated as follows:

For lot A: $10{,}000 \times 0.02 \times 1.00 = Rs\ 200$

Similarly, the cost of refund for other lots is calculated.

Expected cost of refund is:

$$200 \times 0.25 + 500 \times 0.30 + 1{,}000 \times 0.20 + 1{,}500 \times 0.20 + 2{,}000 \times 0.05 = Rs\ 670$$

Expected cost of inspection is:

$$600 \times 0.25 + 600 \times 0.30 + 600 \times 0.20 + 600 \times 0.20 + 600 \times 0.05 = Rs\ 600$$

Since the cost of refund is more than the cost of inspection, the plant should be shut down for inspection. Also, EVPI = EOL of inspection = Rs 170.

**IN-TEXT QUESTIONS**

4. The expected monetary value criterion is used for decision making under risk. True/False

5. The difference between the highest the lowest EMV is said to be EVPI. True/False

6. The payoff due to equally likely criterion of decision making is same as minimum…….

## 5.5 DECISION TREE

A decision tree can graphically represent any issue that can be expressed in a decision table. The different decision-alternatives and the order of events are graphically represented by ***decision trees*** as tree branches. Similar to a network, a decision tree is made up of nodes (or points) and arcs (or lines). They include decision (choice) nodes and states of nature (chance) nodes when building a tree diagram. Thses nodes are depicted by following symbols:

□ **A decision point (or node).** Branches (arcs) coming from the decision point (nodes) denote all decision alternatives available to the decision maker at that point. The decision-maker must choose just one of these alternatives.

○ **Situation of uncertainity** (or an outcome node or event point). Arcs emanating from an outcome node denote all outcomes that could occur at that node. Only one of these possibilities will come true.

These occurrences, which may indicate customer demand or other factors, are not entirely within the decision maker's control. The primary benefit of a tree diagram is that a following act (referred to as a second act) to the occurrence of each event may also be portrayed. In the tree diagram, the outcome (payoff) for each act-event combination may be shown at the extremities of each branch. The following decision tree diagram is displayed:

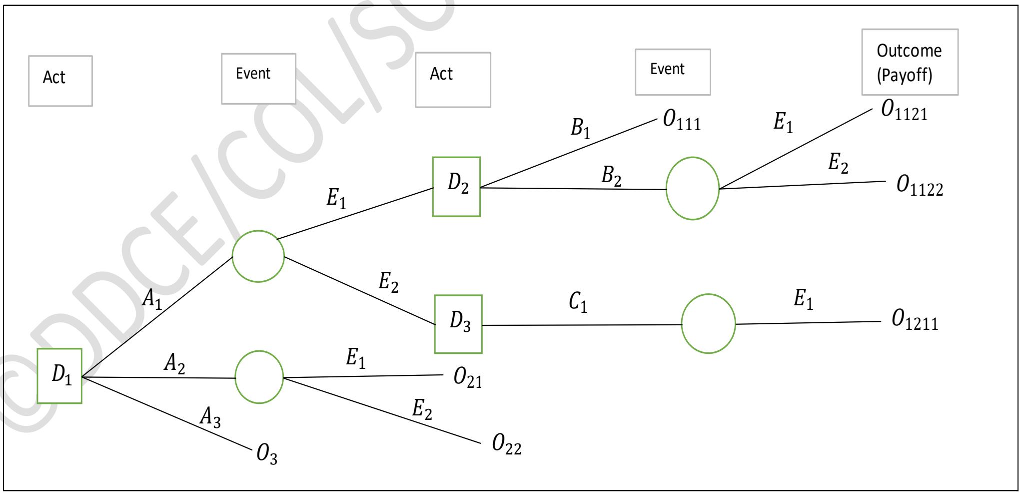



*Fig: A decision Tree Diagram*

$O_{1121}$ represents the payoff of the act event combination $A_1$-$E_1$-$B_2$-$E_1$.

*Folding back* or *rolling back* a decision tree is the process of analysing a decision tree to find the best course of action. By working our way back to the first decision node, we start with the payoffs (i.e., the right extreme of the tree). In folding back the decision tree, we use the following two rules:

- Using the probability of each possible outcome at that node and the payoffs associated with those outcomes, we compute the expected payoff at each outcome node.
- We choose the alternative that produces the better expected payout at each decision node. If the expected payoffs are profits, we choose the alternative with the highest value. In contrast, we choose the alternative with the smallest value if the expected payoffs are costs.

Thus, in a decision-tree, the decision-maker specifies each act-event sequence's potential alternatives, events, and payoff values, along with their probabilities. With the help of this, he may calculate expected payoff values and, as a result, the EMV (expected monetary value) of each act.

When making judgments in scenarios with multiple stages and decisions that are all dependent on one another, a decision tree is a very helpful tool. The computation of EMV for each of the tree's main branches constitutes the contemporary method for decision tree analysis. When the EMV for a particular path has been established, these values become the conditional expected payoffs for their corresponding branches.

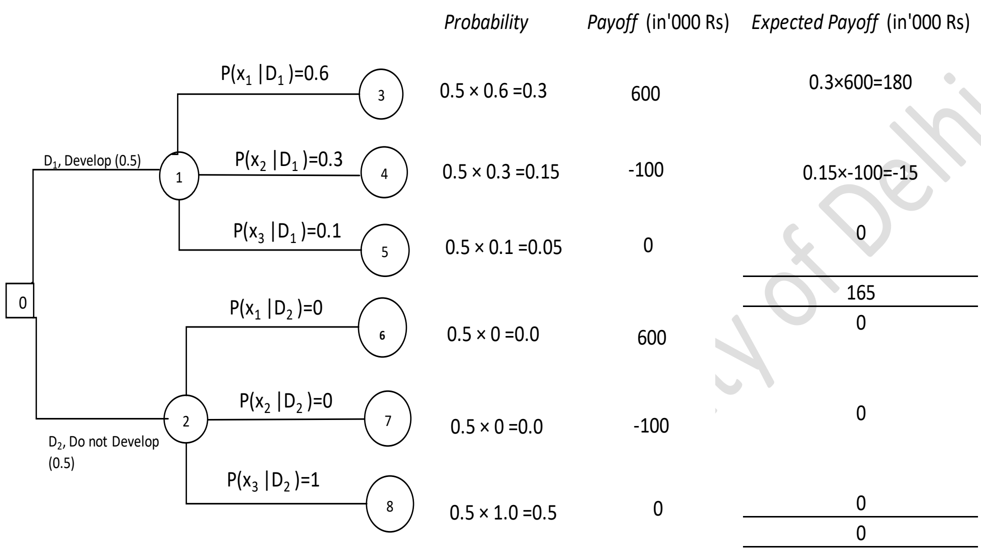
**Example 9:** You are given the following estimates concerning a Research and Development programme:

| Decision $D_i$ | Probability of Decision $D_i$ Given Research R $P(D_i \mid R)$ | Outcome Number | Probability of Outcome $x_i$ given $D_i$ $P(x_i \mid D_i)$ | Payoff Value of Outcome, $x_i$ (Rs'000) |
|---|---|---|---|---|
| **Develop** | 0.5 | 1 | 0.6 | 600 |
| | | 2 | 0.3 | -100 |
| | | 3 | 0.1 | 0 |
| **Do not develop** | 0.5 | 1 | 0.00 | 600 |
| | | 2 | 0.00 | -100 |
| | | 3 | 1.00 | 0 |

Construct and evaluate the decision tree diagram for the above data. Show your workings for evaluation.

**Solution:** The decision tree of the given problem along with necessary calculations is shown in figure 9.1.

**Figure 9.1**



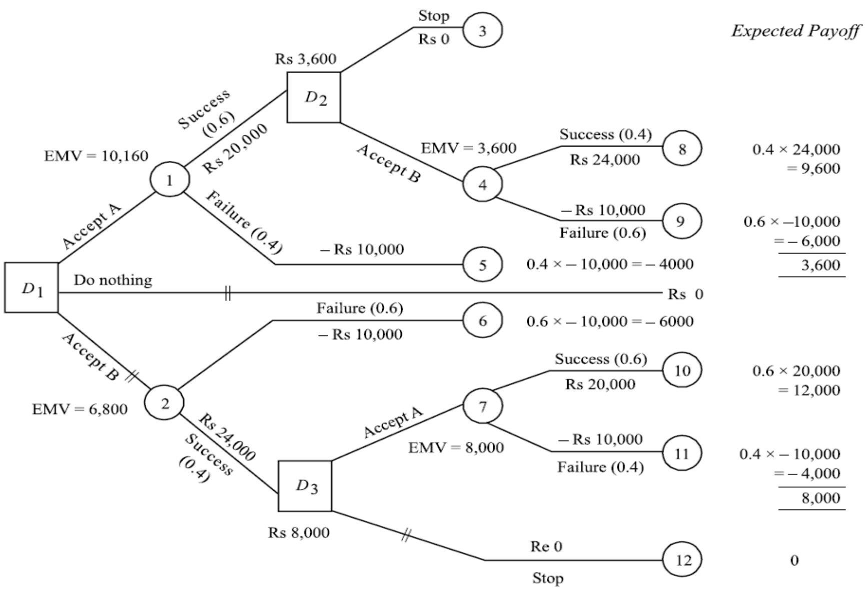

**Example 10** A businessman has two independent investment portfolios A and B, available to him, but he lacks the capital to undertake both of them simultaneously. He can either choose A first and then stop, or if A is not successful, then take, B or vice versa. The probability of success of A is 0.6, while for B it is 0.4. Both investment schemes require an initial capital outlay of Rs 10,000 and both return nothing if the venture proves to be unsuccessful. Successful completion of A will return Rs 20,000 (over cost) and successful completion of B will return Rs 24,000 (over cost). Draw a decision tree in order to determine the best strategy.

**Solution:** The decision tree based on the given information is shown in Fig. 10.1. The evaluation of each chance node and decision is given in Table 12.

**Table 12: Evaluation of Decision and Chance Nodes**

| Decision Point | | Outcome | Probability | Conditional Value (Rs) | Expected Value |
|---|---|---|---|---|---|
| $D_3$ | (i) Accept *A* | Success | 0.6 | 20,000 | 12,000 |
| | | Failure | 0.4 | −10,000 | − 4,000 |
| | | | | | 8,000 |
| | (ii) Stop | – | – | – | 0 |
| $D_2$ | (i) Accept *B* | Success | 0.4 | 24,000 | 9,600 |
| | | Failure | 0.6 | −10,000 | − 6,000 |
| | | | | | 3,600 |
| | (ii) Stop | – | – | – | 0 |
| $D_1$ | (i) Accept *A* | Success | 0.6 | 20,000 + 3,600 = 23,600 | 14,160 |
| | | Failure | 0.4 | −10,000 | − 4,000 |
| | | | | | 10,160 |
| | (ii) Accept *B* | Success | 0.4 | 24,000 + 8,000 = 32,000 | 12,800 |
| | | Failure | 0.6 | −10,000 | − 6,000 |
| | | | | | 6,800 |
| | (iii) Do nothing | – | – | – | 0 |

**Figure 10.1**




Since the EMV = Rs 10,160 at node *D*1 is highest, therefore the best strategy is to accept course of action A first and if A is successful, then accept B.

**IN-TEXT QUESTIONS**

7. A **………..** provides a graphical representation of the various decision-alternatives.

8. The process by which a decision tree is analyzed to identify the optimal decision is referred to as………

## 5.7 SUMMARY

The topic of decision analysis, which is an analytical and systematic method of analysing decision making, is introduced in this chapter. We begin by outlining the procedures involved in decision-making under two different conditions: (1) uncertainty and (2) risk. We use criteria like maximax, maximin, criterion of realism, equally likely, and minimax regret to determine the optimum options for decision-making when faced with ambiguity. We examine the calculation and application of the expected monetary value (EMV), expected opportunity loss (EOL), and expected value of perfect knowledge for decision-making under risk (EVPI). For more complex issues requiring sequential decision-making, decision trees are employed. Here, we calculate the expected value of sample data (EVSI).

## 5.8 GLOSSARY

**Decision Alternative**:- A course of action or a strategy that can be chosen by a decision maker.

**Decision Table:-** A table in which decision alternatives are listed down the rows and outcomes are listed across the columns. The body of the table contains the payoffs. Also known as a *payoff table*.

**Outcome:-** An occurrence over which a decision maker has little or no control. Also known as a state-of-nature.

## 5.9 ANSWERS TO IN-TEXT QUESTIONS

| | |
|---|---|
| 1. Alternative | 5. False |
| 2. Optimism | 6. EOL criterion |
| 3. Minimax regret | 7. Decision Tree |
| 4. True | 8. Folding back |

## 5.10 SELF-ASSESSMENT QUESTIONS

11. What techniques are used to solve decision-making problems under uncertainity? Which technique results in an optimistic decision?
12. State the meanings of EMV and EVPI.
13. A manufacturer manufactures a product, of which the principal ingredient is a chemical X. At the moment, the manufacturer spends Rs 1,000 per year on supply of X, but there is a possibility that the price may soon increase to four times its present figure because of a worldwide shortage of the chemical. There is another chemical Y, which the manufacturer could use in conjunction with a third chemical Z, in order to give the same effect as chemical X. Chemicals Y and Z would together cost the manufacturer Rs 3,000 per year, but their prices are unlikely to rise. What action should the manufacturer take? Apply the maximin and minimax criteria for decision-making and give two sets of solutions. If the coefficient of optimism is 0.4, then find the course of action that minimizes the cost.
14. The manager of a flower shop promises its customers delivery within four hours on all flower orders. All flowers are purchased on the previous day and delivered to Parker by 8.00 am the next morning. The daily demand for roses is as follows.

| | | | | |
|---|---|---|---|---|
| Dozens of roses : | 70 | 80 | 90 | 100 |
| Probability : | 0.1 | 0.2 | 0.4 | 0.3 |

The manager purchases roses for Rs 10 per dozen and sells them for Rs 30. All unsold roses are donated to a local hospital. How many dozens of roses should Parker order each evening to maximize its profits? What is the optimum expected profit?

15. A large steel manufacturing company has three options with regard to production: (*i*) produce commercially (*ii*) build pilot plant (*iii*) stop producing steel. The management has estimated that their pilot plant, if built, has 0.8 chance of high yield and 0.2 hance of low yield. If the pilot plant does show a hight yield, management assigns a probability of 0.75 that the commercial plant will also have a high yield. If the pilot plant shows a low yield, there is only a 0.1 chance that the commercial plant will show a high yield. Finally, management's best assessment of the yield on a commercial-size plant without building a pilot plant first has a 0.6 chance of high yield. A pilot plant will cost Rs. 3,00,000. The profits earned under high and low yield conditions are Rs. 1,20,00,000 and – Rs. 12,00,000 respectively. Find the optimum decision for the company.

## 5.11 REFERENCES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## 5.12 REFERENCES & SUGGESTED BOOKS


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**LESSON 6**

# PROJECT SCHEDULING

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## STRUCTURE

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## 6.1 LEARNING OBJECTIVES

After completing this chapter, you will be able to:

- Understand how to plan, monitor, and control projects using PERT/CPM.

- Determine earliest start, earliest finish, latest start, latest finish, and slack times for each activity.
- Understand the impact of variability in activity times on the project completion time.
- Develop resource loading charts to plan, monitor, and control the use of various resources during a project.
- Understand the trade-cost trade-offs procedure.

## 6.2 INTRODUCTION

Have you ever overseen a significant event? You might have served as the prom committee chair or the board chair for the graduation ceremony in high school. You might have led your team during the introduction of a new product, the planning of a facility expansion, or the implementation of enterprise resource planning. Even if you have never managed people in such circumstances, you have undoubtedly had your own personal projects to contend with, such as writing a paper, moving to a new apartment, applying to college, or selling a house. As a volunteer, you may have overseen the annual function, the elementary school picnic, or the river clean-up project. How did you plan your day's events? Most of your projects, did they finish on time? How did you handle unforeseen circumstances? Did you finish your work on time and on budget? All of these are crucial components of project management. Project managers with expertise are essential assets for organisations since they handle projects frequently.

The listing of activities, deliverables, and milestones within a project constitutes scheduling in project management. An activity's start and end dates, duration, and resources are typically included in a schedule. Successful time management requires effective project scheduling, especially for firms that provide professional services.

A project involves many interrelated activities (or tasks) that must be completed on or before specified time limit, in a specified sequence (or order) with specified quality and minimum cost of using resources such as personnel, money, material, facilities and/or space.

In this lesson we mainly focus on creating and manging schedule. This cover project scheduling with known activity times using well known techniques-PERT and CPM, scheduling with uncertain activity and trade cost trade-offs.

## 6.3 SCHEDULING WITH KNOWN ACTIVITY TIMES

Managers usually must plan, manage, and supervise projects that involve a variety of separate jobs or tasks completed by several departments and individuals. These projects are typically so large or intricate that management frequently struggles to remember every element important to the plan, schedule, and development of the project. In these situations, both the critical path method (CPM) and the programme evaluation and review technique (PERT) have proven to be very helpful.

To aid in the planning and scheduling of the US Navy's massive Polaris Nuclear Submarine Missile programme, which involved thousands of actions, a research team created PERT in 1956–1958. The team's goal was to build and plan the Polaris missile system as efficiently as possible.

The team's goal was to effectively plan and build the Polaris CPM, which was created between 1956 and 1958 by the E.I. DuPont Company and Remington Rand Corporation virtually simultaneously. The organisation set out to create a method for keeping track of chemical plant upkeep.

A wide range of projects can be planned, scheduled, and managed using PERT and CPM:

- Research and development of new products and processes
- Construction of plants, buildings, and highways
- Maintenance of large and complex equipment
- Design and installation of new systems

Project managers are responsible for planning and coordinating the numerous tasks or activities in these kinds of projects to ensure that everything is finished on time.

### 6.3.1 PERT versus CPM

The primary difference between PERT and CPM is in the way the time needed for each activity in a project is estimated. In PERT, each activity has three-time estimates that are combined to determine the expected activity completion time and its variance.

PERT is considered a *probabilistic* technique; it allows us to find the probability that the entire project will be completed by a specific due date. In PERT analysis emphasis is given on the completion of a task rather than the activities required to be performed to complete a task. Thus, PERT is also known as *an event-oriented technique*. PERT is used for one-time projects that involve activities of non-repetitive nature (i.e. activities that may never have been performed before), where completion times are uncertain.

In contrast, CPM is a *deterministic* approach. It estimates the completion time of each activity using a single time estimate. This estimate, called the *standard* or *normal* time, is the time we estimate it will take under typical conditions to complete the activity. In some cases, CPM also associates a second time estimate with each activity. This estimate, called the *crash time*, is the shortest time it would take to finish an activity if additional funds and resources were allocated to the activity. CPM is used for completing of projects that involves activities of repetitive nature.

### 6.3.2 Critical Path Analysis

The objective of critical path analysis is to predict the project's overall duration and give starting and finishing durations to every activity involved. This makes it easier to compare the project's actual progress to its projected completion date.

The expected duration of an activity is estimated from the duration of individual activities, which may be determined uniquely (in the case of CPM) or may entail three-time estimates (in the case of PERT). The following elements need to be understood in order to establish the project scheduling.

i. Total completion time of the project.
ii. Earlier and latest start time of each activity.
iii. Critical activities and critical path.
iv. Float for each activity.

*Notations*:

$E_i$ = Earliest occurrence time of an event, *i*. This is the latest time for an event to occur when all the preceding activities have been completed, without delaying the entire project.

$L_i$ = Latest allowable time of an event, *i*. This is the latest time at which an event can occur without causing a delay in project's completion time.

$ES_{ij}$ = Early starting time of an activity (*i, j*).

$LS_{ij}$ = Late starting time of an activity (*i, j*).

$EF_{ij}$ = Early finishing time of an activity (*i, j*).

$LF_{ij}$ = Late finishing time of an activity (*i*, *j*).
$t_{ij}$ = duration of an activity (*i*, *j*).

There should only be one start event and one finish event in a project schedule. The other events are numbered consecutively with integer 1, 2,…, n, such that *i*<*j* for any two events *i* and *j* connected by an activity, which starts at *i* and finishes at *j*.

These schedule for each activity is created using a two-pass approach that includes a forward pass and a backward pass. The earliest times ($ES_{ij}$ and $EF_{ij}$) are determined during the forward pass. The latest times ($LS_{ij}$ and $LF_{ij}$) are determined during the backward pass.

### 6.3.2.1 Forward Pass Method (For Earliest Event Time)

According to this method, calculations start at the first event, let's say 1, move through the events in increasing order of the event numbers, and finally stop at the last event, let's say N. Each event's *earliest occurrence time* (E), as well as the earliest start and end times for each activity that starts there, are determined. The project's earliest probable completion time is determined by the event N's earliest occurrence time when calculations cease at that point.

The procedure can be summed up as follows:

1. Set the earliest occurrence time of initial event 1 to zero. That is, $E_1$ = 0, for *i* = 1.

2. Calculate the earliest start time for each activity that begins at event *i* (= 1). This is equal to the earliest occurrence time of event, *i* (tail event). That is: $ES_{ij} = E_i$ , for all activities (*i*, *j*) starting at event *i*.

3. Calculate the earliest finish time of each activity that begins at event *i*. This is equal to the earliest start time of the activity plus the duration of the activity. That is: $EF_{ij} = ES_{ij} + t_{ij} = E_i + t_{ij}$ , for all activities (*i*, *j*) beginning at event *i*.

4. Proceed to the next event, say *j*; *j* > *i*.

5. Calculate the earliest occurrence time for the event *j*. This is the maximum of the earliest finish times of all activities ending into that event, that is, $E_j$ = Max $\{EF_{ij}\}$ = Max $\{E_i + t_{ij}\}$, for all immediate predecessor activities.

6. If $j = N$ (final event), then earliest finish time for the project, that is, the earliest occurrence time $E_N$ for the final event is given by $E_N$ = Max { $EF_{ij}$} = Max { $E_N$ – 1 + $t_{ij}$ }, for all terminal activities

#### 6.3.2.2 Backward Pass Method (For Latest Allowable Event Time)

The computations in this technique start with the final event N, move through the events in decreasing sequence of event numbers, and finally arrive at the first event 1. Each event's *latest occurrence time* (L), as well as the most recent start and completion times for each activity that is ending there, are determined. Up until the initial occurrence, the process is repeated.

The following is a summary of the process:

1. Set the latest occurrence time of last event, $N$ equal to its earliest occurrence time (known from forward pass method). That is, $L_N = E_N$ , $j = N$.

2. Calculate the latest finish time of each activity which ends at event $j$. This is equal to latest occurrence time of final event. That is: $LFij = L_i$, for all activities ($i, j$) ending at event $j$.

3. Calculate the latest start times of all activities ending at $j$. This is obtained by subtracting the duration of the activity from the latest finish time of the activity. That is: $LF_{ij} = L_j$ and $LS_{ij} = LF_{ij} – t_{ij} = L_j – t_{ij}$, for all activity ($i, j$) ending at event $j$.

4. Proceed backward to the event in the sequence, that decreases $j$ by 1.

5. Calculate the latest occurrence time of event $i$ ($i < j$). This is the minimum of the latest start times of all activities from the event. That is: $L_i$ = Min {$LS_{ij}$} = Min {$L_j – t_{ij}$ }, for all immediate successor activities.

6. If $j = 1$ (initial event), then the latest finish time for project, i.e. latest occurrence time $L_1$ for the initial event is given by: $L_1$= Min {$LS_{ij}$} = Min{$L_j$ – *1* – $t_{ij}$ }, for all immediate successor activities.

#### 6.3.2.3 Float (Slack) of an Activity and Event

The amount of time that a non-critical activity or event can be postponed or prolonged without extending the overall project completion schedule is known as the float (slack) or free time. Finding the amount of slack time, or spare time, that each activity has is easy once we have determined the earliest and latest timings for all activities. Slack is the amount of time an activity may be postponed without causing the project as a whole to lag. In a project, there are three different sorts of floats for each non-critical activity.

**(a) *Total float*:** This is the amount of time that an activity may be put off until all activities that came before it were finished as soon as possible and all activities that followed it could be put off until the latest time that was permitted.

For each non-critical activity (*i*, *j*) the total float is equal to the latest allowable time for the event at the end of activity *minus* the earliest time for an event at the beginning of the activity *minus* the activity duration. Mathematically,

Total Float $(TF_{ij}) = (L_j - E_i) - t_{ij} = LS_{ij} - ES_{ij} = LF_{ij} - EF_{ij}$

**(b) *Free float*:** This is the amount of time that each non-critical activity's completion time can be pushed back without impacting its immediately succeeding activities. The amount of free float time for a non-critical activity (*i*, *j*) is computed as follows:

Free Float $(FF_{ij}) = (E_j - E_i) - t_{ij}$

$$= Min \{ES_{ij}, for\ all\ immediate\ successor\ of\ activity\ (i,j)\} - EF_{ij}$$

**(c) *Independent float*:** This is the length of time that any non-critical activity $(i, j)$ can be delayed without affecting the completion times of the activities that come before or after it. Each non-critical activity's independent float time is calculated mathematically as follows:

Independent Float $(IF_{ij}) = (E_j - L_i) - t_{ij} = \{ES_{ij} - LS_{ij}\} - t_{ij}$

Independent float values that are negative are regarded as zero.

### 6.3.2.4 Critical Path

Certain activities in any project are called critical activities because delay in their execution will cause further delay in the project completion time. All activities having zero total float value are identified as critical activities, i.e., L = E.

The critical path is the sequence of critical activities between the start event and end event of a project. This is critical in the sense that if execution of any activity of this sequence is delayed, then completion of the project will be delayed. A critical path is shown by a thick line or double lines in the network diagram. The length of the critical path is the sum of the individual completion times of all the critical activities and define the longest time to complete the project. The critical path in a network diagram can be identified as below:

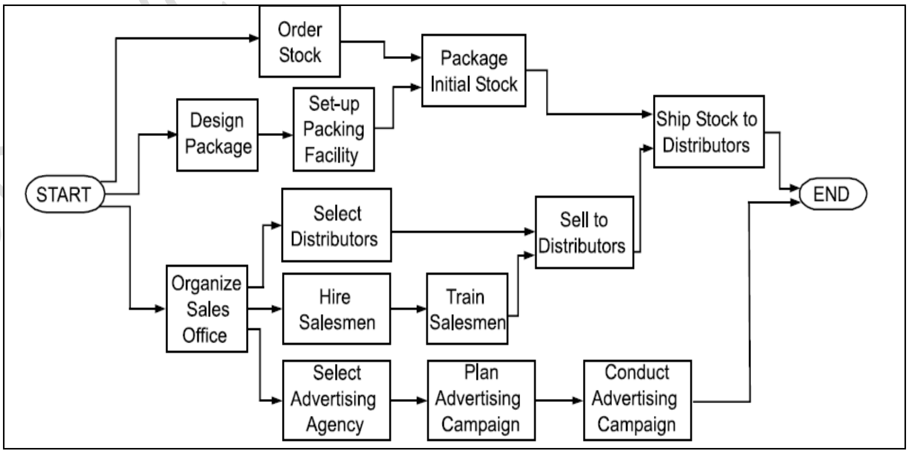
i. If $E_i$ value and $L_j$ value for any tail and head events is equal, then activity (*i*, *j*) between such events is referred as critical, i.e., $E_i = L_i \ and \ E_j = L_j$.

ii. On critical path $E_j - E_i = L_j - L_i = t_{ij}$.

**Example 1:** An established company has decided to add a new product to its line. It will buy the product from a manufacturing concern, package it, and sell it to a number of distributors that have been selected on a geographical basis. Market research has already indicated the volume expected and the size of sales force required. The steps shown in the following table are to be planned.

| *Activity* | *Description* | *Duration (days)* | *Predecessors* |
|---|---|---|---|
| A | Organize sales office | 6 | - |
| B | Hire salesman | 4 | A |
| C | Train salesman | 7 | B |
| D | Select advertising agency | 2 | A |
| E | Plan advertising campaign | 4 | D |
| F | Conduct advertising campaign | 10 | E |
| G | Design package | 2 | - |
| H | Setup packaging facilities | 10 | G |
| I | Package initial stocks | 6 | J, H |
| J | Order stock from manufacturer | 13 | - |
| K | Select distributors | 9 | A |
| L | Sell to distributors | 3 | C, K |
| M | Ship stocks to distributors | 5 | I, L |

The precedence relationship among these activities are shown in the following figure.



As the figure shows, the company can begin to organize the sales office, design the package, and order the stock immediately. Also, the stock must be ordered and the packing facility must be set up before the initial stocks are packaged.

(a) Draw an arrow diagram for this project.

(b) Indicate the critical path.

(c) For each non-critical activity, find the total and free float.

**Solution:** (a) The arrow diagram for the given project, along with *E*-values and *L*-values, is shown in Fig.1. Determine the earliest start time – *Ei* and the latest finish time – *Lj* for each event by proceeding as follows:

*Forward Pass Method*

$E_1 = 0$

$E_2 = E_1 + t_{1,2} = 0 + 6 = 6$

$E_3 = E_1 + t_{1,3} = 0 + 2 = 2$

$E_4 = \text{Max}\,\{E_i + t_{i,4}\} = \underset{i=1,3}{\text{Max}}\,\{E_1 + t_{14};\ E_3 + t_{34}\} = \text{Max}\,\{0 + 13,\ 2 + 10\} = 13$

$E_5 = E_2 + t_{2,5} = 6 + 4 = 10$

$E_6 = \underset{i=2,5}{\text{Max}}\,\{E_i + t_{i,6}\} = \text{Max}\,\{E_2 + t_{2,6};\ E_5 + t_{5,6}\} = \text{Max}\,\{6 + 9;\ 10 + 7\} = 17$

$E_7 = E_2 + t_{2,7} = 6 + 2 = 8$

$E_8 = E_7 + t_{7,8} = 8 + 4 = 12$

$E_9 = \underset{i=4,6}{\text{Max}}\,\{E_i + t_{i,9}\} = \text{Max}\,\{E_4 + t_{4,9};\ E_6 + t_{6,9}\} = \text{Max}\,\{13 + 6;\ 17 + 3\} = 20$

$E_{10} = \underset{i=8,9}{\text{Max}}\,\{E_i + t_{i,10}\} = \text{Max}\,\{E_8 + t_{8,10};\ E_9 + t_{9,10}\} = \text{Max}\,\{12 + 10;\ 20 + 5\} = 25.$

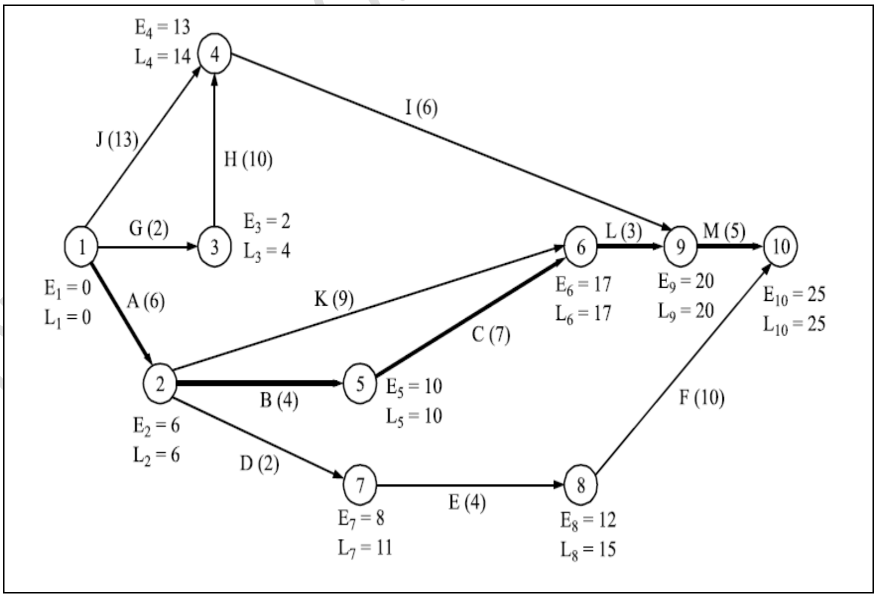



**Figure 1: Network Diagram**

*Backward Pass Method*

$L_{10} = E_{10} = 25$

$L_8 = L_{10} - t_{8,10} = 25 - 10 = 15$

$L_6 = L_9 - t_{6,9} = 20 - 3 = 17$

$L_4 = L_9 - t_{4,9} = 20 - 6 = 14$

$L_2 = \underset{j=5,6,7}{\text{Min}} \{L_j - t_{2,j}\}$

$= \text{Min} \{ L_5 - t_{2,5};\ L_6 - t_{2,6};\ L_7 - t_{2,7}\}$

$= \text{Min} \{10 - 4;\ 17 - 9;\ 11 - 2\} = 6$

$L_9 = L_{10} - t_{9,10} = 25 - 5 = 20$

$L_7 = L_8 - t_{7,8} = 15 - 4 = 11$

$L_5 = L_6 - t_{5,6} = 17 - 7 = 10$

$L_3 = L_4 - t_{3,4} = 14 - 10 = 4$

$L_1 = \underset{j=2,3,4}{\text{Min}} \{L_j - t_{1,j}\}$

$= \text{Min} \{L_2 - t_{1,2};\ L_3 - t_{1,3};\ L_4 - t_{1,4}\}$

$= \text{Min} \{6 - 6;\ 4 - 2;\ 14 - 13\} = 0$

(b) The critical path in the network diagram (Fig.1) has been shown. This has been done by double lines by joining all those events where *E*-values and *L*-values are equal. The critical path of the project is: 1 – 2 – 5 – 6 – 9 – 10 and critical activities are *A, B, C, L* and *M*. The total project completion time is 25 weeks.

(c) For each non-critical activity, the total float and free float calculations are shown in Table1.

**Table 1: Calculation of Floats**

| Activity | Duration | Earliest Time | | Latest Time | | Float | |
|---|---|---|---|---|---|---|---|
| $(i, j)$ | $(t_{ij})$ | Start $(E_i)$ | Finish $(E_i + t_{ij})$ | Start $(L_j - t_{ij})$ | Finish $L_j$ | Total $(L_j - t_{ij}) - E_i$ | Free $(E_j - E_i) - t_{ij}$ |
| 1 – 3 | 2 | 0 | 2 | 2 | 4 | 2 | 0 |
| 1 – 4 | 13 | 0 | 13 | 1 | 14 | 1 | 0 |
| 2 – 6 | 9 | 6 | 15 | 8 | 17 | 2 | 2 |
| 2 – 7 | 2 | 6 | 8 | 9 | 11 | 3 | 0 |
| 3 – 4 | 10 | 2 | 12 | 4 | 14 | 2 | 1 |
| 4 – 9 | 6 | 13 | 19 | 14 | 20 | 1 | 1 |
| 7 – 8 | 4 | 8 | 12 | 11 | 15 | 3 | 0 |
| 8 – 10 | 10 | 12 | 22 | 15 | 25 | 3 | 3 |

**IN-TEXT QUESTIONS**

12. The objective of the project scheduling is to minimize total project cost. True / False
13. The CPM is used for completing the projects that involves activities of repetitive nature. True / False
14. PERT is referred to as an activity-oriented technique. True / False
15. _____________is the time-consuming job or task that is a key subpart of the total project.

## 6.4 SCHEDULING WITH UNCERTAIN ACTIVITY TIMES

We used the CPM technique, which assumes that all activity times are known and fixed constants, to find all earliest and latest times to date as well as the related critical path(s). In other words, activity times are constant. However it is possible that other factors will affect how quickly a task is completed. PERT was developed to handle projects where the time duration for each activity is not known with certainty but is a random variable that is characterized by $\beta$-distribution. To estimate the parameters 'mean and variance' of the $\beta$-distribution three-time estimates for each activity are required to calculate its expected completion time. The necessary three-time estimates are listed below.

i. ***Optimistic time*** $(t_o \text{ or } a)$**:** The shortest possible time (duration) in which an activity can be performed assuming that everything goes well.
ii. ***Pessimistic time*** $(t_p \text{ or } b)$**:** The amount of time needed to complete a task in the worst conceivable circumstances. However, natural disasters like earthquakes, floods, and the like are not included under such circumstances.
iii. ***Most likely time*** $(t_m \text{ or } m)$**:** The amount of time needed to finish a task, if it were repeated numerous times under the same circumstances. Of course, the completion time would happen the most frequently (i.e. model value).

The $\beta$-distribution is not necessarily symmetric; the degree of skewness depends on the location of the $t_m$ to $t_o$ and $t_p$. The range of $t_o$ and $t_p$ is assumed to enclose every possible duration of the activity.

Expected time of an activity $(t_e) = \frac{t_o+4t_m+t_p}{6}$

and variance of activity time, $\sigma_i^2 = \left(\frac{t_p-t_o}{6}\right)^2$.

The variance of the overall critical path's time is calculated by aggregating the variances of the various critical activities if the duration of the activities is a random variable. Suppose $\sigma_c$ is the standard deviation of the critical path. Then

$$\sigma_c^2 = \sum \sigma_i^2 \ \ and \ \sigma_c = \sqrt{\sum \sigma_i^2}$$

### 6.4.1 Estimation of Project Completion Time

There is a potential that the project's scheduled completion time will vary because of the unknown activity completion time. As a result, the decision-maker must be aware of the probability that the specified time will be achieved. Using the central limit theorem, the normal distribution can be used to approximate the probability distribution of completion times for an event. Thus, the probability of completing the project on the schedule time, $T_s$ is given by:

$$prob\left(Z = \frac{T_S - T_e}{\sigma_i}\right)$$

where, $T_e$ = expected completion time of the project

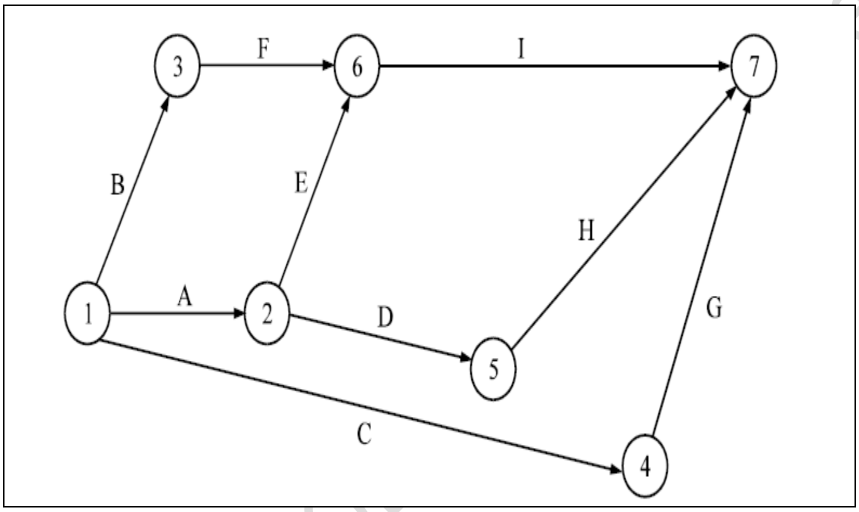
$Z$ = number of standard deviations, the scheduled completion time is away from the mean time.

$\sigma_i{}^2 = \sigma_1{}^2 + \sigma_2{}^2 + \cdots + \sigma_n{}^2$ is the sum of variances of critical activities.

The computation of $T_s$ enables a decision-maker to make certain commitments, knowing the degree of risk. The expected completion time $T_e$ of the project is obtained by adding the expected time of each critical activity.

**Example 2:** The following network diagram represents activities associated with a project:

| Activities | : | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|---|
| Optimistic time, $t_0$ | : | 5 | 18 | 26 | 16 | 15 | 6 | 7 | 7 | 3 |
| Pessimistic time, $t_p$ | : | 10 | 22 | 40 | 20 | 25 | 12 | 12 | 9 | 5 |
| Most likely time, $t_m$ | : | 8 | 20 | 33 | 18 | 20 | 9 | 10 | 8 | 4 |




Determine the following:

(a) Expected completion time and variance of each activity

(b) The earliest and latest expected completion times of each event.

(c) The critical path.

(d) The probability of expected completion time of the project if the original scheduled time of completing the project is 41.5 weeks.

(e) The duration of the project that will have 95 per cent chance of being completed.

**Solution:** Calculations for expected completion time (*te*) of an activity and variance (σ2), using
following formulae are shown in Table 3.

$$(t_e) = \frac{t_o + 4t_m + t_p}{6} \text{ and, } \sigma_i^2 = \left(\frac{t_p - t_o}{6}\right)^2.$$

(b) The earliest and latest expected completion time for all events considering the expected completion time of each activity are shown in Table 3.

**Table 3**

| Activity | $t_o$ | $t_p$ | $t_m$ | $t_e = \frac{1}{6}(t_o + 4t_m + t_p)$ | $\sigma^2 = [\frac{1}{6}(t_p - t_o)]^2$ |
|---|---|---|---|---|---|
| 1 – 2 | 5 | 10 | 8 | 7.8 | 0.696 |
| 1 – 3 | 18 | 22 | 20 | 20.0 | 0.444 |
| 1 – 4 | 26 | 40 | 33 | 33.0 | 5.429 |
| 2 – 5 | 16 | 20 | 18 | 18.0 | 0.443 |
| 2 – 6 | 15 | 25 | 20 | 20.0 | 2.780 |
| 3 – 6 | 6 | 12 | 9 | 9.0 | 1.000 |
| 4 – 7 | 7 | 12 | 10 | 9.8 | 0.694 |
| 5 – 7 | 7 | 9 | 8 | 8.0 | 0.111 |
| 6 – 7 | 3 | 5 | 4 | 4.0 | 0.111 |

*Forward Pass Method*

$E_1 = 0$

$E_2 = E_1 + t_{1,2} = 0 + 7.8 = 7.8$

$E_3 = E_1 + t_{1,3} = 0 + 20 = 20$

$E_4 = E_1 + t_{1,4} = 0 + 33 = 33$

$E_5 = E_2 + t_{2,5} = 7.8 + 18 = 25.8$

$E_6 = \text{Max}\{E_i + t_{i,6}\} = \text{Max}\{E_2 + t_{2,6}; E_3 + t_{3,6}\}$

$= \text{Max}\{7.8 + 20; 20 + 9\} = 29$

$E_7 = \text{Max}\{E_i + t_{i,7}\}$

$= \text{Max}\{E_4 + t_{4,7}; E_5 + t_{5,7}; E_6 + t_{6,7}\}$

$= \text{Max}\{33 + 9.8 ; 25.8 + 8; 29 + 4\} = 42.8$

*Backward Pass Method*

$L_7 = E_7 = 42.8$

$L_6 = L_7 - t_{6,7} = 42.8 - 4 = 38.8$

$L_5 = L_7 - t_{5,7} = 42.8 - 8 = 34.8$

$L_4 = L_7 - t_{4,7} = 42.8 - 9.8 = 33$

$L_3 = L_6 - t_{3,6} = 38.8 - 9 = 29.8$

$L_1 = \text{Min}\{L_j - t_{1j}\}$

$= \text{Min}\{L_2 - t_{1,2}; L_3 - t_{1,3}; L_4 - t_{1,4}\}$

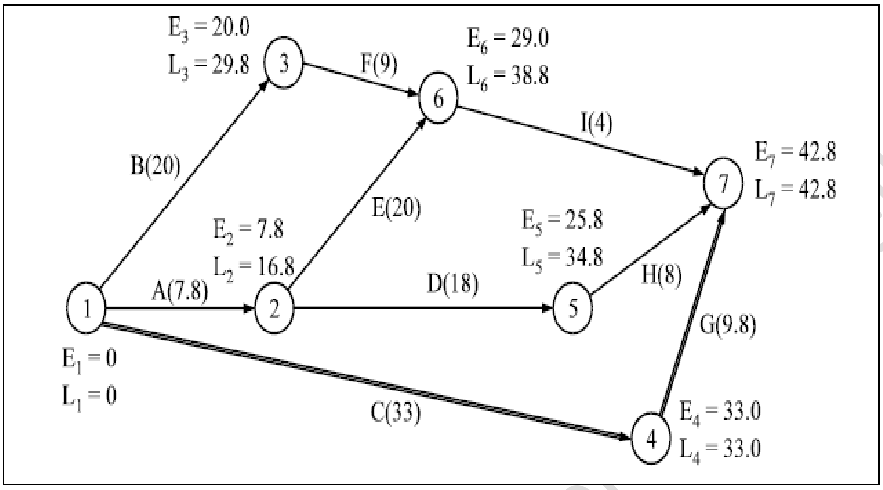
$= \text{Min}\{16.8 - 7.8 ; 29.8 - 20 ; 33 - 33\} = 0$

$L_2 = \text{Min}\{L_j - t_{2,j}\}$

$= \text{Min}\{L_5 - t_{2,5}; L_6 - t_{2,6}\}$

$= \text{Min}\{34.8 - 18 ; 38.8 - 20\} = 16.8$

The *E*-value and *L*-values are shown in Fig. 2.




(c) The critical path is shown by thick line in Fig. 2 where *E*-values and *L*-values are the same. The critical path is: 1 – 4 – 7 and the expected completion time for the project is 42.8 weeks.

(d) Expected length of critical path, $T_e = t_C + t_G$ = 33 + 9.8 = 42.8 weeks (Project duration).

Variance of critical path length, $\sigma^2 = \sigma_C^2 + \sigma_G^2$ = 5.429 + 0.694 = 6.123 weeks.

Since $T_S$ = 41.5, $T_e$ = 42.8 and $\sigma$ = 6.123 = 2.474, the probability of meeting the schedule time is given by: $prob\left(Z \le \frac{T_S - T_e}{\sigma_i}\right) = P\left(Z \le \frac{41.5-42.8}{2.474}\right) = P(Z \le -0.52) = 0.5 - 0.1952$

$$= 0.3048\ (from\ normal\ distribution\ table)$$

Thus, the probability that the project can be completed in less than or equal to 41.5 weeks is 0.3048. In other words, the probability that the project will get delayed beyond 41.5 weeks is 0.6952.

Given that $prob\left(Z \le \frac{T_S - T_e}{\sigma}\right) = 0.95$

But $Z_{0.95}$= 1.64, from normal distribution table. Thus,

$$1.64 = \frac{T_S - 42.8}{2.474}\ or\ T_S = 46.85\ weeks.$$

## IN-TEXT QUESTIONS

5. Beta probability distribution is often used in computing the expected activity completion times and variances in networks. True / False
6. The shortest possible time (duration) in which an activity can be performed assuming that everything goes well is _______________.
7. The amount of time that is expected to complete the activity is called ____________________.

## CASE STUDY

### Krishna Mills

Krishna Mills made the decision to construct a new feed mill in order to improve its production capacity. The project was divided up into various tasks, some of which had to be finished before others could begin. The activities, as well as the anticipated times for each, are listed in Exhibit 1 as "decided upon by management and the precedence relationships." To save as much crucial time as possible while putting the new mill into service, the management sought to move the schedule as far in advance as possible. The mills' president remarked, "If we can get rolling, every week spared is worth Rs 70,000 in lost contribution." Several construction tasks could be accelerated. For instance, by working extra hours, the company's architects could design the new plant in 10 weeks rather than the 12 weeks they had initially planned. The mills will have to pay an extra Rs 25,000 for each week that is advanced due to this advancement. The following table displays the weekly crash cost as well as the maximum amount that each activity could crash. The independent business that was a possible contractor for one of the project's key duties, building the plant, had already been approached by the president of mills. Krishna Mills expected to complete the remaining tasks either directly or via its representatives. . The management had discussed a few bonus and penalty provisions with the mill contractors. One of them was that the mills would pay contractors an extra Rs 75,000 for each week the facility was finished before the allotted 10 weeks.

| *Activity* | *Description Time (weeks)* | *Expected Activities* | *Precedent Time (weeks)* | *Minimum (Rs/week)* | *Crash* |
|---|---|---|---|---|---|
| *A* | Degin plant | 12 | – | 10 | 24,000 |
| *B* | Select plant site | 8 | *A* | 18 | –11 |
| *C* | Select plant builder | 6 | *A* | 11 | 3,000 |
| *D* | Select operating personnel | 13 | *A* | 13 | –11 |
| *E* | Prepare the building site | 4 | *B, C* | 4 | –11 |
| *F* | Make or buy mill equipment | 101 | *C* | 8 | 30,000 |
| *G* | Prepare mill operations manual | 6 | *C* | 4 | 1500 |
| *H* | Build the plant | 101 | *E, F* | 6 | 75,000 |
| *I* | Train plant operators | 8 | *D, G* | 8 | –11 |
| *J* | Test the plant | 8 | *H, I* | 8 | –11 |
| *K* | Obtain a production licence | 4 | *J* | 4 | –11 |

The management of Krishna Mills is interested in learning which operations would crash and how to schedule its employees.

## 6.5 TRADE-COST TRADE-OFFS

The initial creators of CPM gave the project manager the choice to allocate resources to tasks in order to speed up project completion. The option to shorten activity times must consider the increased expenses involved, as more resources (such as additional employees, overtime, etc.) typically raise project costs. In essence, the decision that the project manager must make entails exchanging decreased activity time for increased project cost.

The first key concept for this approach is that of *crashing.*

### 6.5.1 Project Crashing

It is usual for a project manager to encounter one or both of the following circumstances while overseeing a project: Both the projected project completion date and the project's timeline are behind schedule. In either case, some or all of the ongoing tasks must be expedited in order to complete the project by the target deadline. Crashing is the process of reducing the length of a project in the most affordable way possible. Additionally, extending an activity's duration past its usual point (cost-efficient) may raise the expense of carrying out that action. For the sake of simplicity, it is assumed that the relationship between an activity's normal time and cost as well as crash time and cost is linear. Therefore, by calculating the relative change in the cost per unit change in time, the crash cost per unit of time may be determined.

### 6.5.2 Time-Cost Trade-Off Procedure

When all essential tasks are accomplished in accordance with schedule, crashing begins, and it ends when all essential tasks have crashed. The process of determining time-cost trade-offs for project completion can be summed up as follows:

Step 1: Determine the normal project completion time and associated critical path.

Step 2: Identify critical activities and compute the cost slope for each of these by using the relationship

$$Cost\ slope = \frac{Crash\ cost - Normal\ cost}{Normal\ time - Crash\ time}$$

The values of cost slope for critical activities indicate the direct extra cost required to execute an activity per unit of time.

Step 3: For reducing the total project completion time, identify and crash an activity time on the critical path with lowest cost slope value to the point where

i. another path in the network becomes critical, or
ii. the activity has been crashed to its lowest possible time.

Step 4: If the critical path under crashing is still critical, return to step 3. However, if due to crashing of an activity time in step 3, other path(s) in the network also become critical, then identify and crash the activity(s) on the critical path(s) with the minimum joint cost slope.

Step 5: Terminate the procedure when each critical activity has been crashed to its lowest possible time. Determine total project cost corresponding to different project durations.

**Example 3:** The data on normal time, cost and crash time and cost associated with a project are shown in the following table.

| Activity | Normal | | Crash | |
|---|---|---|---|---|
| | Time (weeks) | Cost (Rs) | Time (weeks) | Cost (Rs) |
| 1-2 | 3 | 300 | 2 | 400 |
| 2-3 | 3 | 30 | 3 | 30 |
| 2-4 | 7 | 420 | 5 | 580 |
| 2-5 | 9 | 720 | 7 | 810 |
| 3-5 | 5 | 250 | 4 | 300 |
| 4-5 | 0 | 0 | 0 | 0 |
| 5-6 | 6 | 320 | 4 | 410 |
| 6-7 | 4 | 400 | 3 | 470 |
| 6-8 | 13 | 780 | 10 | 900 |
| 7-8 | 10 | 1,000 | 9 | 1,200 |
| | | 4,220 | | |

Indirect cost is Rs 50 per week.
(a) Draw the network diagram for the project and identify the critical path.
(b) What are the normal project duration and associated cost?
(c) Find out the total float associated with non-critical activities.
(d) Crash the relevant activities and determine the optimal project completion time and cost.

**Solution:** (a) The network for normal activity times is shown in fig 3. The critical path is: $1-2-5-6-7-8$ with a project completion time of 32 weeks.

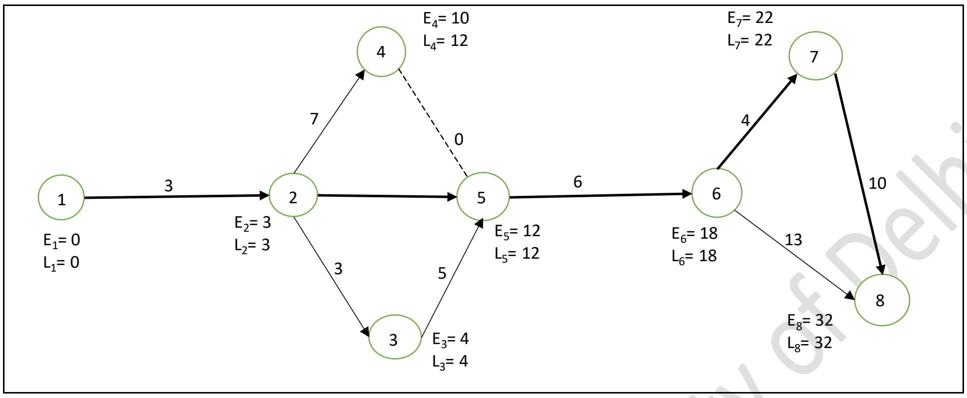



**Figure 3: Network Diagram**

(b) The normal total project cost associated with normal project duration of 32 weeks is as follows:

Total cost = Direct normal cost + Indirect cost for 32 weeks

= 4,220 + 50 × 32 = Rs 5,820

(c) Calculations for total float associated with non-critical activities are shown in table 4.

**Table 4: Total Float**

| Activity | Total Float $(L_j - E_i) - t_{ij}$ |
|---|---|
| 2-3 | (7 - 3) - 3 = 1 |
| 2-4 | (12 - 3) - 7 = 2 |
| 3-5 | (12 - 6) - 5 = 1 |
| 4-5 | (12 - 10) - 0 = 2 |
| 6-8 | (32 - 18) - 13 = 1 |

(d) For critical activities, crash cost-slope is given in table 5.

**Table 5: Crash Cost Slope**

| Critical Activity | Crash Cost per Week (Rs) |
|---|---|
| 1-2 | $\frac{400-300}{3-2} = 100$ |
| 2-5 | $\frac{810-720}{9-7} = 45$ |
| 5-6 | $\frac{410-320}{6-4} = 45$ |
| 6-7 | $\frac{470-400}{4-3} = 70$ |
| 7-8 | $\frac{1200-1000}{10-9} = 200$ |

The minimum value of crash cost per week is for activity 2 – 5 and 5 – 6. Hence, crashing activity 2 – 5 by 2 days from 9 weeks to 7 weeks. But the time should only be reduced by 1 week otherwise another path $1-2-3-5-6-7-8$ become a parallel path. Network, as shown in fig 4, is developed when it is observed that new project time is 31 weeks and the critical path are $1-2-5-6-7-8$ and $1-2-3-5-6-7-8$.

With crashing of activity 2 – 5, the crashed total project cost becomes:

Crashed total cost = Total direct normal cost + Increased direct cost due to crashing of activity (2 – 5) + Indirect cost for 31 weeks

$$= 4{,}220 + 1 \times 45 + 50 \times 31 = 4{,}265 + 1{,}550 = \text{Rs } 5{,}815$$

For revised network shown in fig 4, new possibilities for crashing critical activities are listed in table 6.

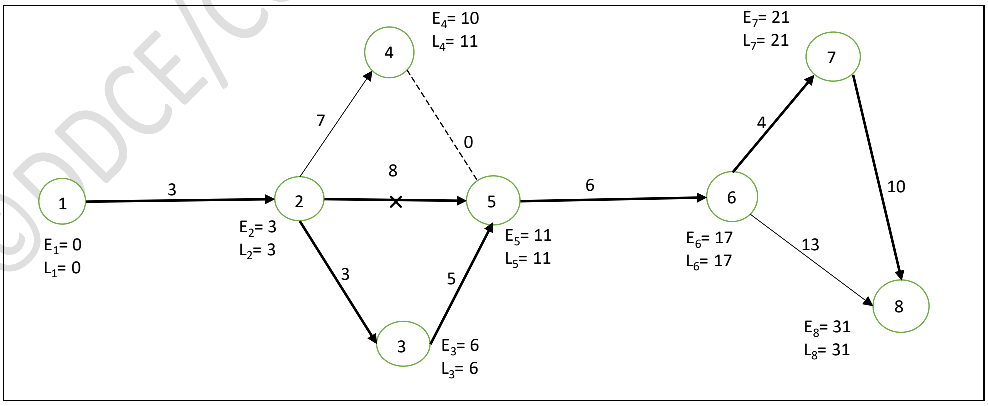



**Figure 4: Network Diagram**

**Table 6: Crash Cost Slope**

| Critical Activity | Crash Cost per Week (Rs) |
|---|---|
| 1-2 | $\frac{400-300}{3-2}=100$ |
| 2-5 | $\times\ (Crashed)$ |
| 2-3 | $0\ (Crashing\ is\ not\ required)$ |
| 3-5 | $\frac{300-250}{5-4}=50$ |
| 5-6 | $\frac{410-320}{6-4}=45$ |
| 6-7 | $\frac{470-400}{4-3}=70$ |
| 7-8 | $\frac{1200-1000}{10-9}=200$ |

Since crashed cost slope for activity 5 – 6 is minimum, its time may be crashed by 2 weeks from 6 weeks to 4 weeks. The updated network diagram is shown in fig 5.

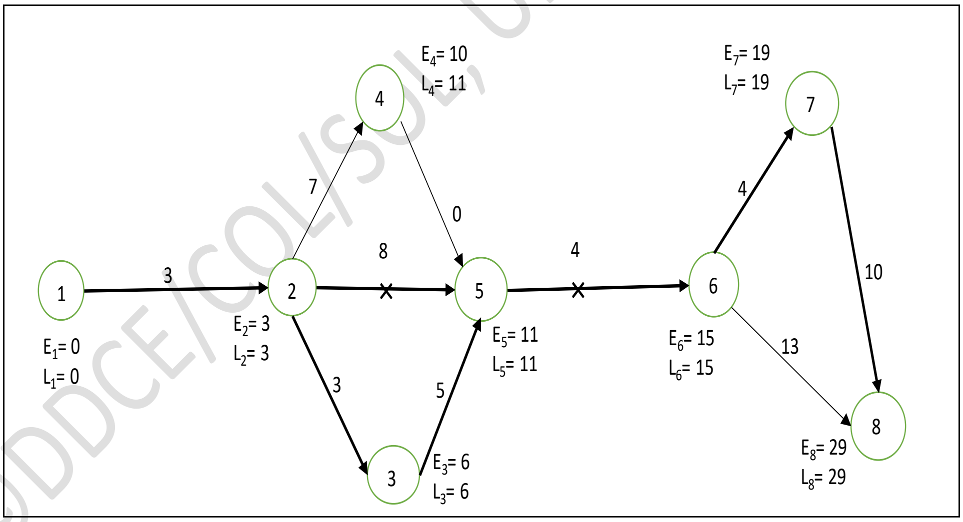



**Figure 5: Network Diagram**

It may be noted in fig 5, that the critical paths shown in fig 4 remain unchanged because activity 5 – 6 is common in both. With crashing of activity 5 – 6 by 2 weeks, the crashed total cost becomes:

Crashed total cost = Total direct normal cost + Increased direct cost due to crashing of activity (5 – 6) + Indirect cost for 29 weeks

$$= 4{,}220 + (1 \times 45 + 2 \times 45) + 50 \times 29 = \text{Rs } 5{,}805$$

For revised network given in fig 5, new possibilities for crashing in the critical paths are listed in table 7.

**Table 7: Crash Cost Slope**

| Critical Activity | Crash Cost per Week (Rs) |
|---|---|
| 1-2 | $\frac{400-300}{3-2} = 100$ |
| 2-3 | $0\ (Crashing\ is\ not\ required)$ |
| 2-5 | $\times\ (Crashed)$ |
| 5-6 | $\times\ (Crashed)$ |
| 6-7 | $\frac{470-400}{4-3} = 70$ |
| 7-8 | $\frac{1200-1000}{10-9} = 200$ |



The further crashing 6 – 7 activity time from 4 weeks to 3 weeks will result in increased direct cost than the gain due to reduction in project time. Hence, terminate crashing. The optimal project duration is 29 weeks with associated cost of Rs 5,805 as shown in table 8.

**Table 8: Crashing Schedule of Project**

| *Project Duration (weeks)* | *Crashing Activity and Weeks* | *Direct Cost (Rs)* | | | *Indirect Cost (Rs)* | *Total Cost (Rs)* |
|---|---|---|---|---|---|---|
| | | *Normal* | *Crashing* | *Total* | | |
| 32 | – | 4,220 | – | 4,220 | $32 \times 50 = 1{,}600$ | 5,820 |
| 31 | 2 - 5(1) | 4,220 | $1 \times 45 = 45$ | 4,265 | $31 \times 50 = 1{,}550$ | 5,815 |
| 29 | 5 - 6(2) | 4,220 | $45 + 2 \times 45 = 135$ | 4,355 | $29 \times 50 = 1{,}450$ | 5,805 |
| 28 | 6 - 7(1) | 4,220 | $135 + 1 \times 70 = 205$ | 4,425 | $28 \times 50 = 1{,}400$ | 5,825 |

**IN-TEXT QUESTIONS**

8. The process of shortening the duration of a project in the least expensive manner possible is called ____________________.

9. In time-cost trade-off function analysis the:

a) cost decreases linearly as time increases b) cost at normal time is zero
c) cost increases linearly as time decreases d) none of the above

## 6.6 SUMMARY

PERT (Program Evaluation and Review Technique) and CPM (Critical Path Method) have been widely used to help project managers plan, schedule, and manage their projects ever since they were developed in the late 1950s.

When using PERT/CPM, a project is first broken down into its separate activities, their immediate predecessors are noted, and the time of each activity is estimated. The creation of a project network to display this information is the next phase.

PERT/CPM produce project scheduling data, such as the earliest start time, latest start time, and slack for each activity. Also, it outlines the actions that must be completed in a specific order in order to avoid delays in project completion. Given that the critical path is the longest path through the project network, if all activities proceed according to plan, the length of the critical path establishes the project's duration.

Yet, because there is frequently a great deal of ambiguity over how long an activity will really last, it is challenging for all activities to continue on schedule. By collecting three different types of estimates (most likely, optimistic, and pessimistic) for the length of each activity, the three-estimate approach in PERT addresses this dilemma. The mean and variance of the probability distribution for this duration are approximately determined using this information. The likelihood that the project will be completed by the deadline can then be roughly calculated.

The project manager can analyse the impact on total cost of adjusting the expected duration of the project to various alternative values using the time-cost trade-offs approach in CPM. The time and cost for each action while it is carried out normally and when it is completely crashed are the statistics required for this activity (expedited).

## 6.7 GLOSSARY

- **Activity: -** A job or task that consumes time and is a key subpart of a total project.
- **Critical Activities: -** Critical activities have zero slack time.
- **Critical Path Method (CPM): -** A deterministic network technique that is similar to PERT but uses only one time estimate. CPM is used for monitoring budgets and project crashing.
- **Event: -** A point in time that marks the beginning or ending of an activity.



## 6.8 ANSWERS TO IN-TEXT QUESTIONS

| | |
|---|---|
| 1. False | 6. Optimistic time |
| 2. True | 7. Most likely time |
| 3. False | 8. Crashing |
| 4. Activity | 9. Option a |
| 5. True | |

## 6.9 SELF-ASSESSMENT QUESTIONS

16. Explain the following term in PERT/CPM
    I. Earliest time
    II. Latest time

III. Total activity time

17. PERT takes care of uncertain duration. How far is this statement correct?

## 6.10 REFERENCES


- Balakrishnan, N., Render, B., Stair, R. M., & Munson, C. (2017). Managerial decision modeling. Upper Saddle River, Pearson Education.
- Hillier, F.& Lieberman, G.J. (2014). *Introduction to operations research* (10th ed.).McGraw-Hill Education.


## 6.11 SUGGESTED READINGS


- Anderson, D., Sweeney, D., Williams, T., Martin, R.K. (2012). *An introduction to management science: quantitative approaches to decision making* (13th ed.). Cengage Learning.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# LESSON 7
# MARKOV PROCESSES

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## STRUCTURE

7.1 Learning Objectives
7.2 Introduction
7.3 Stochastic Process
7.4 State Space
7.5 Classification of Stochastic Process
7.6 Markov Chain
7.7 Transition probability
7.8 Transition probability matrix
7.9 Initial distribution
7.10 Concept for Classification of the states
7.10.1 Accessibility
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7.10.4 Closed set of states
7.10.5 Irreducible & reducible chain
7.10.6 Absorbing state
7.10.7 Periodicity
7.10.8 First visit probability
7.10.9 Mean Passage time
7.10.10 First return probability
7.10.11 Mean recurrence time
7.11 Classification of the states
7.11.1 Recurrent State
7.11.2 Transient State
7.11.3 How to determine whether a state is recurrent or transient

through transition graph
7.12 Some important results
7.13 Basic limit theorem for aperiodic markov chain
7.14 Stationary distribution
7.15 Application areas of markov chain
7.16 Summary
7.17 Glossary
7.18 Answers to In-text Questions
7.19 Self-Assessment Questions
7.20 Suggested Readings

## 7.1 LEARNING OBJECTIVES

After reading the unit, students will be able to

- Define the concepts of stochastic process.
- Describe the terminologies used in stochastic process.
- Explain Markov process.
- Understand the transition probabilities.
- To check whether a state is transient or recurrent.
- Identify the situation where Markov chains can be used in Business.

## 6.2 INTRODUCTION

Andrei Markov was a Russian mathematician who lived from 1856 to 1922. The only subject he did well in was math, and he had a dismal grade point average overall. Later, he was taught the subject by Pafnuty Chebyshev, a mathematics lecturer at the University of Petersburg who is well known for his work in probability theory. Markov first focused on number theory, convergent series, and approximation theory as his three primary scientific disciplines. His most famous research on Markov chains is where the phrase originates, and his first article on the subject appeared in 1906.

The Markov chain is a fundamental mathematical tool for stochastic processes. The Markov Property is the essential idea, according to which some stochastic process predictions can be made more simply by treating the future as independent of the past in light of the process's present state. This is done to make stochastic process future state forecasts simpler to comprehend. This section will explore the principles of Markov chains, explain the different types of Markov Chains, and provide instances of its use in business and finance.

Markov chains are employed to determine the chance of events changing states. We'll use the weather as an example: A sunny day enhances the probability of the next day being sunny by 70% and reduces the probability of it being wet by 30%. There is a 20% chance that tomorrow will be sunny if it rains today, but an 80% probability that it will rain again. This can be summarised in a transition diagram, where each potential state change is shown in Fig. 1 of the diagram.

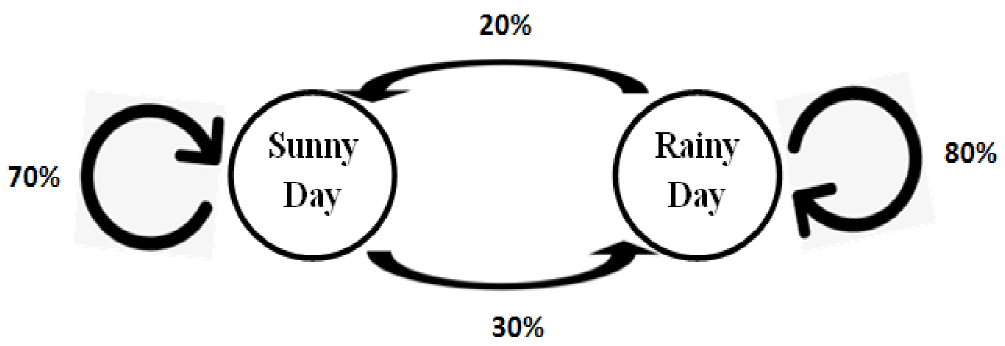



Fig.-1: Transition diagram

## 7.3 STOCHASTIC PROCESS

A stochastic process is one whose outcomes depend on some element of chance. A stochastic or random process is a collection of random variables that is indexed by a mathematical set, which means that each random variable in the stochastic process is specifically linked to an element in the set. The index set is the collection used to index the random variables. In the past, the index set was a subset of the real line, such as I the natural number, which gave the index set a temporal interpretation.

The collection's random variables all draw their values from the same state space, a body of mathematics. The real line, the integers, or the n-dimensional Euclidean space are a few examples of the state space.

{X(t), t ∈T}, defined on some probability space (Ω, F, P), where T is a parameter space, is referred to as a stochastic process. State space refers to the collection of all potential values for all random variables, and states are its constituent parts.

**Example:** $X_1$ = first toss, $X_2$ = second toss, ………, $X_n$ = $n^{th}$ toss.
Then, the collection of random ($X_1$, $X_2$, ……., $X_n$) variables is called stochastic process.

## 7.4 STATE SPACE

The values assumed by a random variable X(t) are called or states and the collection of all possible values forms the state space (S) of the process. If X(t)= i , then we say the process is in state i.

**a) Discrete state process:** This state space is finite or countable for example the non-negative integers {0,1,2,3,….}.

**b) Continuous state process:** This state space contains finite of infinite intervals of the real number line.

## 7.5 CLASSIFICATION OF STOCHASTIC PROCESS

A stochastic process can be classified in different ways for example, by its state space, its index set, or the dependence among the random variable.

**a) Discrete/ Continuous time:** A stochastic process is considered to be in discrete time if the index set has a finite or countable number of elements, such as a finite set of numbers, the set of integers, or the natural numbers. Discrete-time stochastic process is the name given to this particular kind of stochastic process. Time is referred to as continuous and stochastic processes are referred to as continuous - time stochastic processes if the index set of the stochastic process is some interval of the real line.

**b) Discrete/ Continuous state space:** The stochastic process is referred to as a discrete or integer-valued stochastic process if the state space consists of integers or natural numbers. The stochastic process is known as a real-valued stochastic process or a process with continuous state space if the state space is the real line.

## 7.6 MARKOV CHAIN

A sequence of random variables $\{X_n$, where n = 0, 1, 2, 3, …..} with discrete state space is known as markov chain if,

$$P_r(X_n = K \mid X_{n-1} = j, X_{n-2} = j_1, \ldots\ldots, X_0 = j_{n-1}) = P_r(X_n = K \mid X_{n-1} = j) = p_{jk}$$

Example: $X_1 = 0,1$, $X_2 = 0,1$, ………….., $X_n = 0,1$

Then the partial sum of random variable or present value is given by,

$$S_n = X_1 + X_2 + \ldots\ldots\ldots\ldots + X_n = \{0, 1, 2, \ldots\ldots, n\}$$

So, the future value is, $S_{n+1} = X_1 + X_2 + \ldots\ldots\ldots\ldots + X_n + X_{n+1}$

Therefore the markov chain is, $\{S_n, n \geq 1\}$

## 7.7 TRANSITION PROBABILTY

Probability of going from state i to state j is known as transition probability.

1-step probability from n to n-1: $P(X_n = j \mid X_{n-1} = i) = p_{ij}$

2-step probability from n to n-2: $P(X_n = j \mid X_{n-2} = i) = p_{ij}$

n-step probability from 2n to n: $P(X_{2n} = j \mid X_n = i) = p_{ij}$

## 7.8 TRANSITION PROBABILTY MATRIX

Let S be a state space, such that S = {0, 1, 2, …….} then the transition probability matrix is given by,

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ - \\ - \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & - & - \end{array} \\ \begin{bmatrix} p_{00} & p_{01} & p_{02} & - & - \\ p_{10} & p_{11} & p_{12} & - & - \\ p_{20} & p_{21} & p_{22} & - & - \\ -- & -- & -- & - & - \\ -- & -- & -- & - & - \end{bmatrix} \end{array}$$

Where,

$p_{00} + p_{01} + p_{02} + \dots\dots = 1$

$p_{10} + p_{11} + p_{12} + \dots\dots = 1$

**Properties of transition probability matrix:**

i) $p_{ij} \in S \geq 0$
ii) If each row sum is 1, then the matrix is known as stochastic matrix.
iii) If each column sum is 1, then the matrix is known as doubly stochastic matrix.

**Example:** Suppose that the probability of a dry day (state 0) followinga rainy day is 1/3 and probability of a rainy day (state 1) following a dry day is 1/2. If there is a two-state Marcov chain such that $p_{10} = 1/3$ and $p_{01} = 1/2$ and the transition probability matrix (TPM),

$$P = \begin{array}{c} 0 \\ 1 \end{array}\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \quad (\text{columns } 0,\ 1)$$

Given that may 1 is a dry day, find the probability that May 3 is a dry day.

Solution: Given that $X_1$ = May 1 is a dry day.

Probability that $X_3$ = May 3 is a dry day is given by,

$$P(X_3 = 0 \mid X_1 = 0) = p_{00}^{(2)} = p^2$$

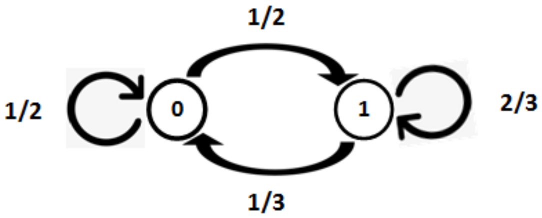
$$P^2 = P.P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = \begin{array}{c} 0 \\ 1 \end{array}\begin{bmatrix} 5/2 & 7/12 \\ 7/18 & 11/18 \end{bmatrix} \quad (\text{columns } 0,\ 1)$$

Therefore, $P(X_3 = 0 \mid X_1 = 0) = 5/12$

**Second method using transition graph:** Let the transition matrix is

$$P = \begin{matrix} & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 1 & 1/3 & 2/3 \end{matrix}$$

Then the transition graph is,



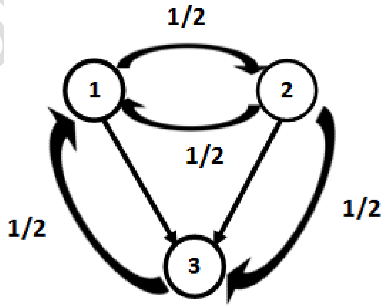

And $p_{00}^{(2)} = p_{01}\, p_{10} + p_{00}\, p_{00} = (1/2)(1/3) + (1/2)(1/2) = 5/12$

**Example:** Consider a markov chain $\{X_n \mid \geq 0\}$ with state space $\{1, 2, 3\}$ and transition matrix,

$$P = \begin{matrix} & 1 & 2 & 3 \\ 1 & 0 & 1/2 & 1/2 \\ 2 & 1/2 & 0 & 1/2 \\ 3 & 1/2 & 1/2 & 0 \end{matrix}$$

Then, find $P(X_3 = 1 \mid X_0 = 0)$

Solution: The transition graph corresponding to the given TPM is,




And $p_{11}^{(3)} = p_{12}\, p_{23}\, p_{31} + p_{13}\, p_{32}\, p_{21} = (1/2)(1/2)(1/2) + (1/2)(1/2)(1/2) = 1/4$

Therefore, $P(X_3 = 1 \mid X_0 = 0) = ¼$

## 7.9 INITIAL DISTRIBUTION

Let the state space is {0, 1, 2, .....}, initial state = 0, then $P(X_0 = i)$ is called initial distribution and

$$P(X_0 = i) = \pi_i$$

**Example:** Let $\{X_n, n \geq 0\}$ be a markov chain with 3 sattes 0, 1, 2 and with transition matrix

$$P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

And initial distribution, $P_r(X_0 = i) = 1/3, i = 0, 1, 2$

Then, find $P(X_3 = 1 \mid X_0 = 1)$ and calculate the joint probability, $P(X_3 = 1, X_1 = 1, X_0 = 2)$

Solution: S = {0, 1, 2}

$$P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

$p_{11}^{(3)} = p_{12}\, p_{22}\, p_{21} + p_{11}\, p_{12}\, p_{21} + p_{10}\, p_{00}\, p_{01}$

$= (1/4)(1/4)(3/4) + (1/2)(1/4)(3/4) + (1/4)(3/4)(1/4)$

$= 3/16$

Therefore, $P(X_3 = 1 \mid X_0 = 1) = 3/16$

Now, $p_{11}^{(2)} = p_{11}\, p_{11} + p_{12}\, p_{21} + p_{10}\, p_{01}$

$= (1/2)(1/2) + (1/4)(3/4) + (1/4)(1/4)$

$= 1/2$

Now, $P(X_3 = 1, X_1 = 1, X_0 = 2) = P(X_3 = 1, X_1 = 1 \mid X_0 = 2)\, P(X_0 = 2)$

$= P(X_3 = 1 \mid X_1 = 1)\, P(X_1 = 1 \mid X_0 = 2)\, P(X_0 = 2)$

$= p_{11}^{(2)} \,.\, p_{21}^{(1)} \,.\, (1/3)$

$= (1/2)(3/4)(1/3)$

$= 1/8$

## 7.10 CONCEPT OF CLASSIFICATION OF STATES

### 7.10.1 Accessibility

If $p_{ij}^{(n)} > 0$, where $n \geq 1$, then state j is accessible from state i.

Let if $p_{01} = 1/2 > 0$, which means that state 1 is accessible from state 0.

And if $p_{10} = 0$, which means that state 0 is not accessible from state 1.

### 7.10.2 Communicating state

Let i,j ∈S such that, $p_{ij}^{(n_1)} > 0$ and $p_{ji}^{(n_2)} > 0$

$\Rightarrow i \leftrightarrow j$

i.e., i and j are communicating states.

### 7.10.3 Communicating class

Let i ∈S then C(i) is called the communicating class such that

$$C(i) = \{ i \in S \mid i \leftrightarrow i \}$$

Let, i, j, k ∈S and $i \leftrightarrow j$, $j \leftrightarrow k$, $k \leftrightarrow i$

Then, C(i) = {i, j, k}

### 7.10.4 Closed set of states

If i and j communicate only with each other, not from other states, then C(i) = {i, j} is called the closed set of states.

### 7.10.5 Irreducible & reducible chain

A markov chain is said to be irreducible if every state communicate with each other, i.e., there is only one communicating class.

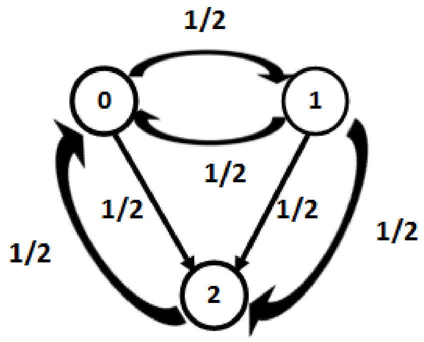
i.e., C(i) = S

otherwise the markov chain is called reducible markov chain.

**Example:** Check whether the given transition matrix is irreducible or reducible for the state space {0, 1, 2}

(i) $$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

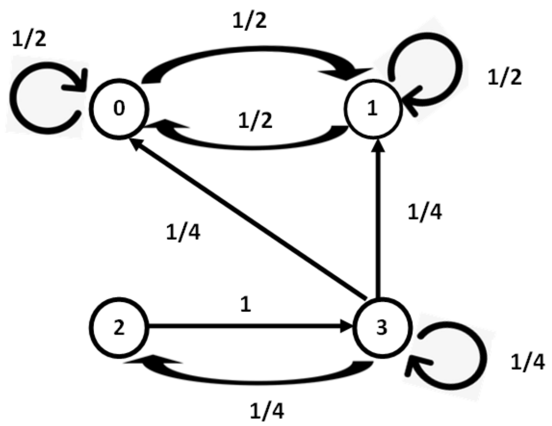
Solution: The transition diagram for the given TPM is,




Obviously, C(0) = {0, 1, 2}
Therefore, the given transition matrix is irreducible.

(ii) $$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

Solution: The transition diagram for the given TPM is,




From the diagram it is clear that,
C(0) = {0, 1}
C(1) = {0, 1}
C(2) = {2, 3}
C(3) = {2, 3}
Therefore, the given transition matrix is reducible.

### 7.10.6 Absorbing state

If for any state i, C(i) has only one element, then the state is called an absorbing state.
Eg.: If C(i) = {i}, then i is called absorbing state.
And if C(j) = {j, k}, then j is not an absorbing state.

**Example:** Find the absorbing states from the following TPM:

$$P = \begin{matrix} & 0 & 1 & 2 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 0 & 0 & 1 \end{matrix}$$

Solution: From the TPM it is clear that,
C(0) = {0, 1}
C(1) = {0, 1}
C(2) = {2}
Here, {2} is absorbing state.

### 7.10.7 Periodicity

The period of the state i ∈ S is defined as d(i) or λ(i) and is given by,

$$d(i) = \gcd\ (n \geq 1 \mid p_{ii}^{(n)} > 0)$$

where n is the number of steps.

**Remarks:**

- If any state has a self loop then its period is 1.
- If d(i) = 1, then the state i is aperiodic state.
- If d(i) > 1, then the state i is periodic.
- If i and j are communicating states then period of i and j will be equal, i.e., d(i) = d(j)

**Example:** find the period of the states from the following transition diagram:
Solution: From the diagram it is clear that,
d(0) = gcd{1, 2, 3, .....} = 1
d(1) = gcd{ 2, 3, 4, .....} = 1

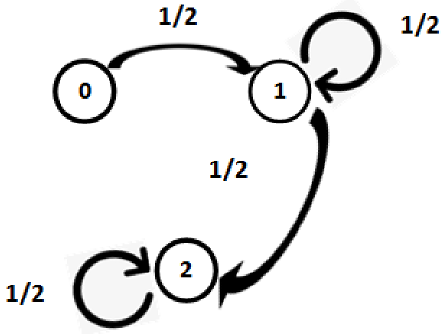
### 7.10.8 First visit probability

The first visit probability is given by,

$$f_{ij}^{(n)} = P\{X_n = j, X_m \neq j, \forall m < n \mid X_0 = i\}$$

i.e., i→j in n-steps but can not visit i in less than n-steps. It must be n-steps only.

**Example:** Find $f_{02}^{(2)}$ from the following transition diagram:




Solution: From the graph, we can write

$f_{02}^{(2)}$ = (1/2)(1/2) = 1/4

### 7.10.9 Mean Passage time

If $f_{ij}^{(n)}$ be the first visit probability, then mean passage time is given by,

$$\mu_{ij} = \sum_{n=1}^{\infty} n f_{ij}^{(n)}$$

### 7.10.10 First return probability

If $f_{ii}^{(n)}$ denotes the probability that, starting from state i, the first return to state i, in n[th] time step,

i.e., $f_{ii}^{(n)} = P\{X_n = i, X_m \neq i, \forall m < n \mid X_0 = i$

The probabilities $f_{ii}^{(n)}$ are known as first return probabilities.

$$f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)}$$

$f_{ii}$ = Probability [Ever return to i | $X_0$ = i]

### 7.10.11 Mean recurrence time

If $f_{ii}^{(n)}$ be the first return probability, then mean recurrence time is given by,

$$\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$$

**Example:** Let $\{X_n, n \geq 0\}$ be a 2-state markov chain with state space S = {0, 1} and transition matrix,

$$P = \begin{matrix} & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 1 & 1/3 & 2/3 \end{matrix}$$

Assouming $X_0 = 0$, find the expected return time to 0.

Solution: We know that, $\mu_{00} = \sum_{n=1}^{\infty} n f_{00}^{(n)} = 1\ f_{00}^{(1)} + 2\ f_{00}^{(2)} + 3\ f_{00}^{(3)} + 4\ f_{00}^{(4)} + 5\ f_{00}^{(5)} + .........$

Now, $f_{00}^{(1)} = p_{00}^{(1)} = 1/2$

$f_{00}^{(2)} = p_{01}p_{10} = (1/2)(1/3) = 1/6$

$f_{00}^{(3)} = p_{01}p_{11}p_{10} = (1/2)(2/3)(1/3) = 1/9$

$f_{00}^{(4)} = p_{01}p_{11}p_{11}p_{10} = (1/2)(2/3)(2/3)(1/3) = 2/27$

$f_{00}^{(5)} = p_{01}p_{11}p_{11}p_{11}p_{10} = (1/2)(2/3)(2/3)(2/3)(1/3) = 4/81$

Therefore,

$\mu_{00} = 1\ (1/2) + 2\ (1/6) + 3\ (1/6)(2/3) + 4\ (1/6)(2/3)^2 + 5\ (1/6)(2/3)^3 + ........$

$= 1/2 + 1/3 + (1/6)(2/3)\ [3 + 4\ (2/3) + 5\ (2/3)^2 + .......]$

Let $S = 3 + 4\ (2/3) + 5\ (2/3)^2 + .......$

$(2/3)S = \quad 3\ (2/3) + 4\ (2/3)^2 + 5\ (2/3)^3 + .......$

Subtracting we get,

$(1/3)S = 3 + (2/3) + (2/3)^2 + (2/3)^3 + .......$

$= 3 + (2/3)[1 + (2/3) + (2/3)^2 + (2/3)^3 + ......]$

$= 3 + (2/3)[1/(1 – (2/3))] = 5$

So, S = 15

Therefore, the expected return time to 0 is $\mu_{00} = 1/2 + 1/3 + (1/6)(2/3)\ [15] = 5/2$

## 7.11 CLASSIFICATION OF STATES

### 7.11.1 Recurrent State

If the first return probability for any state i, $f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, then the state is called a recurrent state.

For any recurrent state, if mean recurrence time, $\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)} < \infty$ (finite), then it is called a positive recurrent state and if $\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)} = \infty$ (infinite), then it is called a null recurrent state.

### 7.11.2 Transient State

If the first return probability for any state i, $f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} < 1$, then the state is called a transient state.

**Example:** consider the chain on states {1, 2, 3, 4} with TPM

$$P = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

Find the transient and recurrent states.

Solution: Here,

$f_{11}^{(1)} = 1/3$

$f_{11}^{(2)} = p_{12}\, p_{21} = (2/3)\,(1) = 2/3$

$f_{11}^{(3)} = p_{12}\, p_{22}\, p_{21} = (2/3)(0)(1) = 0$

Therefore, $f_{11}^{(n)} = 0$, if $n \geq 3$

$$f_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} + \ldots\ldots = 1/3 + 2/3 + 0 + 0 + \ldots\ldots = 1$$

So, state 1 is recurrent state.

Now, $f_{22}^{(1)} = 0$

$f_{22}^{(2)} = p_{21}p_{12} = (1)(2/3) = 2/3$

$f_{22}^{(3)} = p_{21}p_{11}p_{12} = (1)(1/3)(2/3) = 2/9$

$$f_{22} = \sum_{n=1}^{\infty} f_{22}^{(n)} = f_{22}^{(1)} + f_{22}^{(2)} + f_{22}^{(3)} + \ldots\ldots.. = 0 + 2/3 + 2/9 < 1$$

So, state 2 is transient state.

Now, $f_{33}^{(1)} = 1/2$

$f_{33}^{(2)} = p_{31}p_{13} = (1/2)(0) = 0$

$$f_{33} = \sum_{n=1}^{\infty} f_{33}^{(n)} = f_{33}^{(1)} + f_{33}^{(2)} + \ldots.. = 1/2 + 0 < 1$$

So, state 3 is transient state.

Now, $f_{44}^{(1)} = 1/2$

$f_{44}^{(2)} = p_{43}p_{34} = (1/2)(0) = 0$

$$f_{44} = \sum_{n=1}^{\infty} f_{44}^{(n)} = f_{44}^{(1)} + f_{44}^{(2)} + \ldots.. = 1/2 + 0 < 1$$

So, state 4 is transient state.

Hence, state 1 is the recurrent state and states 2, 3, 4 are transient states.

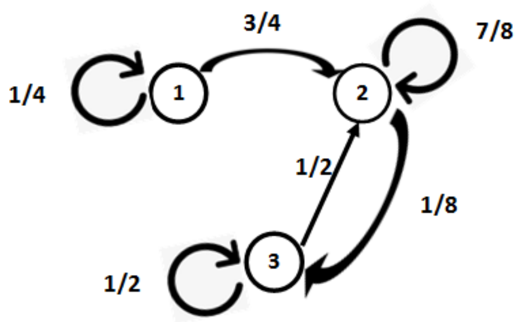
### 7.11.3 How to determine whether a state is recurrent or transient through transition graph

In the transition graph, if we move out from any state and we found that there is no route to come back to the starting state, then the state is called transient state, otherwise it is called a recurrent state.

**Example:** Find the transient and recurrent states from the following TPM:

$$P = \begin{matrix} & 1 & 2 & 3 \\ 1 & 1/4 & 3/4 & 0 \\ 2 & 0 & 7/8 & 1/8 \\ 3 & 0 & 1/2 & 1/2 \end{matrix}$$

Solution: The transition graph of the given TPM is,

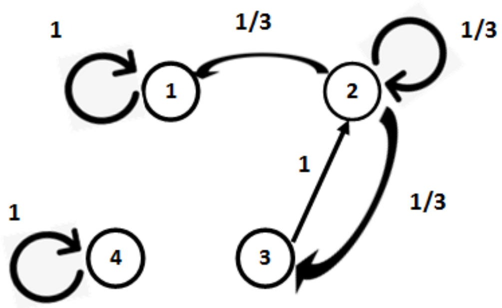



For state 1, if we move out from state 1, we found that there is no route to come back to state 1, therefore state 1 is transient.
For state 2, if we move out from state 2, we found that there is a route to come back to state 2, therefore state 2 is recurrent.
Similarly, for state 3, if we move out from state 3, we found that there is a route to come back to state 3, therefore state 3 is recurrent.

**Example:** consider the chain on states {1, 2, 3, 4} with TPM

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the absorbing, transient and recurrent states using transition graph.
Solution: The transition graph of the given TPM is,




Since, C(1) = {1} and C(4) = {4}, therefore 1 and 4 are absorbing state.
For state 1, if we move out from state 1, we found that there is a route to come back to state 1, therefore state 1 is recurrent.

For state 2, if we move out from state 2, we found that there is no route to come back to state 2, therefore state 2 is transient.
For state 3, if we move out from state 3, we found that there is no route to come back to state 3, therefore state 3 is transient.
Similarly, for state 4, if we move out from state 4, we found that there is a route to come back to state 4, therefore state 4 is recurrent.
Hence, 2 and 3 are transient states and 1 and 4 are recurrent states.

## 7.12 SUMMARY

- If state i is recurrent iff, $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$
- If state i is transient iff, $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$
- If state i is transient then, $p_{ii}^{(n)} \to 0, when \quad n \to \infty$

### IN-TEXT QUESTIONS

16. ________________ are a fundamental part of stochastic processes and are used widely in many different disciplines.
17. $if \sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$
    The state i is called ____________.
    a) Recurrent state b) Transient state
    c) Both of these d) None of these
18. A ___________ irreducible markov chain has all recurrent states.
19. If a TPM is doubly stochastic matrix, then the sum of each column and each row is______ .

**Theorem:** Recurrence and transient are class properties.

Let i→recurrent and C(i) = {i, j}, i.e., i and j are communicating then j→recurrent.

or, if i is transient then j is also transient.

**Remarks:**

- A finite irreducible markov chain has all recurrent states.
- A finite markov chain has atleast one recurrent state.

## 7.13 BASIC LIMIT THEOREM FOR APERIODIC MARKOV CHAIN

Let $\{X_n: n = 1, 2, 3, .....\}$ be a recurrent, irreducible and aperiodic markov chain with transition probability matrix $P = (p_{ij})$, then

$$\lim_{n\to\infty} p_{ij}^{(n)} = \frac{1}{\mu_{ii}}$$

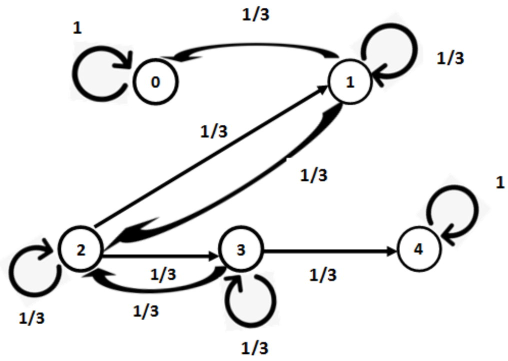
where $\mu_{ii}$ is mean recurrence time for state i.

**Example:** Consider a markov chain with state space {0, 1, 2, 3, 4}. The TPM is given below:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then find $\lim_{n\to\infty} p_{23}^{(n)}$

Solution: The transition graph for the given TPM is,




From the graph is it clear that, 0 and 4 are absorbing states, hence recurrent.

1, 2 and 3 are transient states.

Since, 3 is transient, therefore

$$\lim_{n\to\infty} p_{23}^{(n)} = 0$$

## 7.14 STATIONARY DISTRIBUTION

Consider a markov chain with transition probability $p_{jk}$ and TPM $P = [p_{jk}]$. A probability distribution $\{v_j\}$ is called stationary or invariant for the given chain if

$$v_k = \sum_j v_j p_{jk}$$

Such that,

$$v_j \geq 0, \sum_j v_j = 1$$

Again,

$$v_k = \sum_j v_j p_{jk} = \sum_j \{\sum_i v_i p_{ij}\} = \sum_i v_i p_{ik}$$

And in general,

$$v_k = \sum_i v_i p_{ik}^{(n)}, n \geq 1$$

Let us consider the TPM,

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & - & - \\ p_{21} & p_{22} & p_{23} & - & - \\ p_{31} & p_{32} & p_{33} & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix}$$

Matrix notation, V = VP or, $\pi = \pi P$

Let the set of states ={1, 2, 3, .......}

$$\pi = [\pi_1 \quad \pi_2 \quad \pi_3 \quad ..........]$$

$$\pi = \pi P$$

$$[\pi_1 \quad \pi_2 \quad \pi_3 \quad ..........] = [\pi_1 \quad \pi_2 \quad \pi_3 \quad ..........] \begin{bmatrix} p_{11} & p_{12} & p_{13} & - & - \\ p_{21} & p_{22} & p_{23} & - & - \\ p_{31} & p_{32} & p_{33} & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix}$$

After multiplication, we have

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31} + ......$$

$$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32} + ......$$

$$\pi_3 = \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33} + ......$$

...................................................

...................................................

$$\sum_{i=1}^{n} \pi_i = 1$$

$$\pi_1 + \pi_2 + .... + \pi_n = 1$$

Solving these equations, we can find the stationary distribution.

**Example:** Let S = {1, 2, 3} with TPM,

$$P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$$

Find the stationary distribution.

Solution: We know that,

$$\pi = \pi P$$

$$[\pi_1 \quad \pi_2 \quad \pi_3] = [\pi_1 \quad \pi_2 \quad \pi_3]\begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$$

After multiplication, we have

$$\pi_1 = (1/2)\pi_1 + (3/4)\pi_2 \qquad \text{............ (1)}$$

$$\pi_2 = (1/3)\pi_1 + \pi_3 \qquad \text{............ (2)}$$

$$\pi_3 = (1/6)\pi_1 + (1/4)\pi_2 \qquad \text{............ (3)}$$

And $\pi_1 + \pi_2 + \pi_3 = 1$ ............ (4)

Solving these equations, we have

$$\pi_1 = (3/2)\pi_2$$

$$\pi_3 = (1/2)\pi_2$$

Using values of $\pi_1$ and $\pi_3$ in equation (4), we have

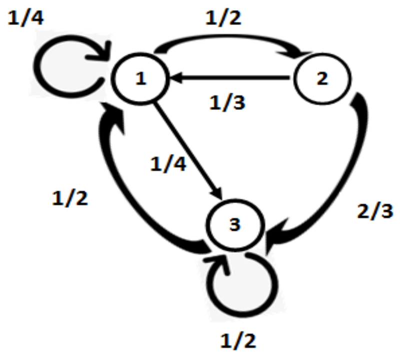
$$(3/2)\pi_2 + \pi_2 + (1/2)\pi_2 = 1$$

$$\pi_2 = 1/3$$

Therefore, $\pi_1 = 1/2$ and $\pi_3 = 1/6$

Hence, the required stationary distribution is, $[\pi_1 \quad \pi_2 \quad \pi_3] = [1/2 \quad 1/3 \quad 1/6]$

**Example:** Consider the following markov chain,



Show that the given chain is irreducible and aperiodic. Also find the stationary distribution for this chain.

Solution: From the transition graph it is clear that,

C(1) = {1, 2, 3}, therefore the chain is irreducible.

Also, 1 and 3 are self loops, therefore d(1) = 1, d(3) = 1

And all the states are communicating, so d(2) = 1

Since for the given markov chain the period of each state is 1, therefore it is aperiodic chain.

Th e TPM for the given chain is,

$$P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

We know that,

$$\pi = \pi P$$

$$[\pi_1 \quad \pi_2 \quad \pi_3] = [\pi_1 \quad \pi_2 \quad \pi_3] \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

After multiplication, we have

$$\pi_1 = (1/4)\pi_1 + (1/3)\pi_2 + (1/2)\pi_3 \qquad \text{............. (1)}$$

$$\pi_2 = (1/2)\pi_1 \qquad \text{............. (2)}$$

$$\pi_3 = (1/4)\pi_1 + (2/3)\pi_2 + (1/2)\pi_3 \qquad \text{............. (3)}$$

And $$\pi_1 + \pi_2 + \pi_3 = 1 \qquad \text{............. (4)}$$

Solving these equations, we have

$$\pi_1 = 2\pi_2$$

$$\pi_3 = (7/3)\pi_2$$

Using values of $\pi_1$ and $\pi_3$ in equation (4), we have

$$2\pi_2 + \pi_2 + (7/3)\pi_2 = 1$$

$$\pi_2 = 3/16$$

Therefore, $\pi_1 = 3/8$ and $\pi_3 = 7/16$

Hence, the required stationary distribution is, $[\pi_1 \quad \pi_2 \quad \pi_3] = [3/8 \quad 3/16 \quad 7/16]$

## IN-TEXT QUESTIONS

20. For any recurrent state, if mean recurrence time, $\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)} < \infty$ (finite), then it is called a ______________.

21. If d(i) = 1, then the state i is ____________.
    a) Transient state
    b) Aperiodic state
    c) Recurrent state
    d) None of these

22. if C(j) = {j, k}, then j is not an ______________.

23. If i and j communicate only with each other, not from other states, then C(i) = {i, j} is called the ____________ of states.



**Example:** Let $\{P_n, n \geq 0\}$ be a sequence of numbers, such that all $n \geq 0$, $P_n > 0$,

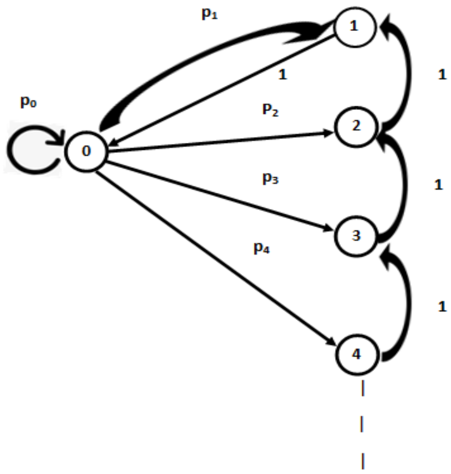
$$\sum_{n=0}^{n} P_n = 1 \quad , \quad \sum_{n=0}^{n} nP_n < \infty$$

Consider the markov chain with S = {0, 1, 2, .......} with TPM

$$P = \begin{bmatrix} p_0 & p_1 & p_2 & - & - \\ 1 & 0 & 0 & - & - \\ 0 & 1 & 0 & - & - \\ 0 & 0 & 1 & - & - \\ - & - & - & - & - \end{bmatrix}$$

Show that the chain in irreducible and positive recurrent.

Solution: The transition diagram for the given TPM is,




From the graph it is clear that,

C(0) = {0, 1, 2, 3, ..........} = S

Therefore, the chain is irreducible and since all the states are communicating.

Also there are paths so that we can start from any state and return to it, therefore all the states are recurrent.

Now, $\mu_{00} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$

$= 1\ f_{00}^{(1)} + 2\ f_{00}^{(2)} + 3\ f_{00}^{(3)} + ..........$

$= 1\ (p_0) + 2\ (p_1) + 3\ (p_2) + ..........$

$= [p_0 + (p_1 + p_2 + p_3 + ......)] + (p_1 + 2p_2 + 3p_3 + ......)$

$= 1 + \sum_{n=1}^{\infty} nP_n$

$\mu_{00} < \infty$

Therefore, the chain is positive recurrent.

## 7.15 APPLICATION AREAS OF MARKOV CHAINS

Markov chains are utilised in a wide range of contexts because they may be created to simulate a variety of real-world processes. These disciplines include speech recognition, search engine algorithms, and the mapping of animal life populations. They are frequently used in economics and finance to forecast macroeconomic events like market crashes and cycles between recession and boom. Predicting asset and option values and estimating credit risks are two other applications. To mimic the randomness in a continuous-time financial market, Markov chains are used. For instance, a stochastic discount factor, which is defined using a Markov chain, determines the price of an item.

## 7.16 SUMMARY

A key idea in stochastic processes is the Markov chain. They can be used to significantly simplify processes that meet the Markov property, which states that a stochastic variable's future state depends only on its current state. This means that understanding the process's past performance won't help with future projections, which naturally minimises the quantity of information that must be taken into account. It is possible to identify specific patterns in a market's prior moves by examining its historical data. Markov diagrams can then be created from these patterns and used to forecast future market movements and the dangers attached to them.

## 7.17 GLOSSARY

- **The Markov chain** - The Markov chain is a fundamental mathematical tool for stochastic processes. The Markov Property is the essential idea, according to which some stochastic process predictions can be made more simply by treating the future as independent of the past in light of the process's present state.
- **A stochastic process** - A stochastic or random process is a collection of random variables that is indexed by a mathematical set, which means that each random variable in the stochastic process is specifically linked to an element in the set.
- **Markov chain** - A sequence of random variables $\{X_n$, where n = 0, 1, 2, 3, …..} with discrete state space is known as markov chain if,
    - $P_r(X_n = K \mid X_{n-1} = j, X_{n-2} = j_1, \ldots\ldots, X_0 = j_{n-1}) = P_r(X_n = K \mid X_{n-1} = j) = p_{jk}$
- **Absorbing state -** If for any state i, C(i) has only one element, then the state is called an absorbing state.
- **Recurrent State -** If the first return probability for any state i, $f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, then the state is called a recurrent state.

- **Transient State -** If the first return probability for any state i, $f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} < 1$, then the state is called a transient state.

## 7.18 ANSWERS TO IN-TEXT QUESTIONS

| | |
|---|---|
| 1. Markov chains | 5. Positive recurrent state |
| 2. Recurrent state | 6. Aperiodic state |
| 3. Finite | 7. Absorbing state |
| 4. 1 | 8. Closed set |

## 7.19 SELF-ASSESSMENT QUESTIONS

1) Consider the markov chain with state space S = {1, 2, 3} with TPM

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Let $\pi = [\pi_1 \quad \pi_2 \quad \pi_3]$ be the stationary distribution of markov chain nd d(1) denotes the period of state 1. Show that d(1) = 1 and $\pi_1 = 1/3$.

2) Consider the markov chain $\{X_n : n \geq 0\}$ on state space S = {0, 1} with TPM P. then

show that if $$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, $\lim_{n\to\infty} P[X_n = i]$ converges for i = 0, 1, but limits depand on initial distribution v.

3) There is a calculator that simply employs the digits 0 and 1. One of these digits is meant to be transmitted through a number of phases. At each stage, though, there is a chance p that the digit that enters will be altered when it exits and a chance q = 1 - p that it won't. Create a Markov chain using the digits 0 and 1 as states to represent the transmission process. What is the transition probabilities matrix? Create a tree now

and assign probabilities based on the assumption that the process starts in state 0 and goes through two transmission stages. What is the likelihood that the machine will eventually create the digit 0 after two stages?

4) Suppose that a man can work as a professional, a skilled worker, or an unskilled worker. Suppose that, among the sons of professionals, 80% work in the field of their fathers' profession, 10% are skilled labourers, and 10% are unskilled labourers. Sons of skilled labourers make up 60% of the labour force, 20% of professionals, and 20% of unskilled workers. In the case of unskilled labourers, 50% of the sons work as such, with 25% of them falling into each of the other two categories. Assuming that every man has at least one son, create a Markov chain by choosing a son at random from each family and following that son's career path through numerous generations. Create the transition probabilities matrix. Calculate the likelihood that a randomly selected untrained labourer's grandson is a professional man.

## 7.20 SUGGESTED READINGS


- B. Sericola (2013). Markov Chains: Theory, Algorithms and Applications. *London: ISTE Ltd and John Wiley & Sons Inc*.
- R. G. Gallager (2013). Stochastic processes: theory for applications. *United Kingdom: Cambridge university press*.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**LESSON 8**

**THEORY OF GAMES**

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## STRUCTURE

8.1 Learning Objectives
8.2 Introduction
8.3 Game Models
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  8.4.1 When saddle point exists
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## 8.1 LEARNING OBJECTIVES

- The students will learn the concept of game theory for decision making in managerial problems.
- It will equip them to know the consequences of interplay and pay-offs with the use of each combination of strategies by the players in the game.
- Students will understand various game models and their solutions to find out the optimal strategies and expected pay-off for the players in the game.

## 8.2 INTRODUCTION

Game Theory helps in decision-making in situations of conflict of interests where two or more rational opponents are competing against each other. The situation of conflicting interests among the opponents is called a *game* and the opponents or the decision-makers in the game are the *players.* Theory of games equips in determining the rules of rational behaviour for the players in the game situation in which the outcomes are dependent on the actions of interdependent players.

Each player in the game has its own set of strategies. A *strategy* is the action taken by the player in various game situations. Each strategy chosen by the player in a game situation leads to outcomes called *pay-offs*. Each player in the game is assumed to be rational which means that a player's preference of strategies is determined by the order of magnitude of the payoffs with each strategy. . The pay-offs of various combinations of strategies by the players are given and known to them but their objections can be different and the outcomes are interdependent on each others' actions. The solution of the game calls for determing the optimal strategies for the players. Each player strives for the optimal strategy. An *optimal strategy* provides the best situation in the game as it involves the maximal pay-off for the players.

Firms competing for market share, labour union striking against management, players in a chess game, companies competing for sales are game situations in real-life. Theory of games helps in addressing such situations of competition and conflicting interests among opponents with an objective of rational decision making with optimal solution for the players.

## 8.3 SCHEDULING WITH KNOWN ACTIVITY TIMES

There are different game models which are classified as follows.

**On the basis of players:**

Game situations where two opponents/players are competing against each other are called *two-person game*. Games which involve more than two players are known as *n-person game,* where it does not necessarily mean that exactly *n persons* are involved in the game but rather the participants can be categoried into *n* mutually exclusive categories and members of these categories have identical interests.

**On the basis of sum of gain and loss:**

Game situations where the sum of gains and losses is equal to zero are called *zero-sum or constant sum games.* Games where the sum of gains and losses is not zero are called *non-zero games.* For example, if the players decide that at the end of the game, the loser would pay Rs. 500 to the winner, then it is called a zero-sum game because the loss of one player is the gain of the other player (sum of gain and loss being equal to zero).

**On the basis of stratgies:**

Game situations where players have a option of choosing from only a finite number of strategies are called *finite games.* Game situations where players have a option of choosing from an infinite number of strategies are called *infinite games.*

In our analysis, we deal with two-person zero-sum games with finite choices for players.

## 8.4 TWO-PERSON ZERO-SUM GAME

A two-person zero-sum game is the one which involves two players with competing interest and gain of one is equal to the loss of another. To illustrate, let's assume there are two companies Alpha Limited (A) and Beta Limited (B) which are competing for the market share. Now, given the total size of the market, gain of market share of one firm would lead to the loss of market share for another. Thus, it is a zero-sum game as sum of gains and losses for both the firms is equal to zero.

Now, let's assume that both the firms are considering four strategies to increase their market share; High advertising, celebrity endorsements, free samples and social media marketing. We assume that currently they have equal market share and further each of the firm can employ only one strategy at a time.

Given the above conditions, 4 × 4 =16 combinations of moves are possible. High advertising by Alpha Limited can be accompanied by high advertising, celebrity endorsements, free samples and social media marketing by Beta Limited. Similarly, celebrity endorsements by Alpha Limited can be accompanied by high advertising, celebrity endorsements, free samples and social media marketing by Beta Limited and so on for further strategies. Each combination of strategy will affect the market share in a particular way giving the pay-offs. For example, high advertising by Alpha Limited and high advertising by Beta Limited will lead to 16 points (implying 16% market share) in favour of Alpha Limited. Similarly, high advertising by Alpha Limited accompanied by celebrity endorsements by Beta Limited leads to 17 points (17% market share in favour of Beta Limited. Similarly, there are pay-offs for each combination of strategies employed by Alpha and Beta.

The pay-offs are shown in the matrix below. The strategies of high advertising, celebrity endorsements, free samples and social media marketing employed by Alpha are given as $a_1$, $a_2$, $a_3$ and $a_4$ and strategies of high advertising, celebrity endorsements, free samples and social media marketing employed by Beta are given as $b_1$, $b_2$, $b_3$ and $b_4$ in the table. Please note the pay-off matrix is drawn from Alpha's viewpoint which means that a positive pay-off means that Alpha has gained the market share at the expense of Beta and the negative pay-offs imply Beta's gain at the expense of Alpha.

| | | **Beta's Strategies** | | | |
|---|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
| **Alpha's strategies** | $a_1$ | 16 | -17 | -8 | 9 |
| | $a_2$ | 6 | 8 | -5 | -13 |
| | $a_3$ | 11 | 9 | 12 | 16 |
| | $a_4$ | 3 | 4 | 9 | 8 |

Now, we need to understand that both the companies are aware of the pay-off matrix but they are not aware of the strategy that the other one will choose. The conservative approach to select the best strategy will be to assume the worst and act accordingly. Thus, with reference to the pay-off matrix, if Alpha Limited chooses strategy $a_1$, it would expect Beta Limited to choose strategy $b_2$, resulting in -17 as the pay-off for Alpha. If Alpha assumes $a_2$, it would expect Beta to select $b_4$, resulting in -13 as the pay-off matrix. Similarly, if Alpha chooses $a_3$, it would lead to 9 as the pay-off as it would expect Beta to select $b_2$ and choosing $a_4$ as the strategy by Alpha would lead to 3 as the pay-off as it would expect Beta to select $b_1$ strategy. We need to keep in mind that both the companies know the pay-off but are unaware of other chosen strategy and are conservative in deciding their strategy based on the pay-offs.

The company Alpha Limited would like to make the best use of the situation by choosing the maximum out of these minimum pay-offs. In other words, it would select the highest of the minimum pay-offs for each of the four strategies. This decision rule is called *maximin* *stretagy*- choosing maximum out of minimum pay-offs. Since, the minimum pay-off for each strategy for Alpha $a_1$, $a_2$, $a_3$ and $a_4$ is -17, -13, 9 and 3 respectively; Alpha Limited would select maximum out of these pay-offs i.e. 9 which is corresponding to strategy $a_3$ (free samples).

Similarly, Beta Limited would also be conservative in its approach. If Beta chooses $b_1$, then it would expect Alpha to choose $a_1$ (maximum advantage for Alpha) resulting in 16 as the pay-off. . If Beta chooses $b_2$, then it would expect Alpha to choose $a_3$ resulting in 9 as the pay-off. Similarly, choosing $b_3$ by Beta would result in a pay-off of 12 as Alpha would be expected to choose $a_3$ strategy and if Beta select $b_4$, then Alpha would be expect to select $a_3$ resulting in 16 as the pay-off. To mimize the advantage to Alpha, Beta would select the strategy that yields the minimum advantage to its competitor. Hence, the decision of Beta Limited will be in accordance to the *minimax strategy*- selecting minimum out of the maximum pay-offs. Since, the maximum pay-off for each strategy for Alpha $a_1$, $a_2$, $a_3$ and $a_4$ is 16, 9, 12 and 16 respectively; Beta Limited would select minimum out of these maximum pay-offs i.e. 9 which is corresponding to strategy $b_2$ (celebrity endorsements).

It should be noted that corresponding to maximin rule for Alpha Limited and minimax rule for Beta Limited, the pay-off is 9. This pay-off is the value of the game which represents the final pay-off to the winner by the losing player. Since, the pay-off is 9, which is drawn from Alpha's point of view, it means the game situation is favourable towards Alpha Limited. If the game value was negative, then it would be favourable towards Beta Limited. The game would said to be fair or equitable if the value of the game was zero. This means it favours none of the players.

Thus, in the above example, Alpha's optimal strategy is $a_3$ (giving free samples) and Beta's optimal strategy is $b_2$ (celebrity endorsements) and the value of the game is 9 which means 9% market share in favour of Alpha Limited. The game situation is favourable towards Alpha.

### 8.4.1 Saddle Point

The point of equilibrium where the maximin value is equal to the minimax value is called *saddle point*. To obtain the saddle pint, we find the row minima (minimum pay off for each row in the pay off matrix) and the column maxima (maximum pay off for each column in the pay off matrix). In case, maximum of row minima is equal to minimum of colum maxima, then the value represents the saddle point. Let's consider our previous example.

| Beta's Strategies | | | | | | |
|---|---|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ | $b_4$ | Row minima |
| Alpha's strategies | $a_1$ | 16 | -17 | -8 | 9 | -17 |
| | $a_2$ | 6 | 8 | -5 | -13 | -13 |
| | $a_3$ | 11 | 9 | 12 | 16 | 9* |
| | $a_4$ | 3 | 4 | 9 | 8 | 3 |
| | Column maxima | 16 | 9* | 12 | 16 | |

In the table, we find the row minima (miminum pay-off for each row) and column maxima (maximum pay-off for each column). As we can see the maximum of row minima (maximun strategy) and minimum of column maxima (minimax strategy) is the same i.e. 9. This is the point of equilibrium (saddle point). It represents the value of the game and implies that Alpha limited will gain 9% market share at the cost of Beta Limited. The game situation is favourable to Alpha Limited.

A game can have more than one saddle points as well for a given problem. Let's consider another example.

| **Beta's Strategies** | | | | | | |
|---|---|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ | $b_4$ | Row minima |
| **Alpha's strategies** | $a_1$ | 16 | -17 | -8 | 6 | -17 |
| | $a_2$ | 6 | 8 | -5 | -13 | -13 |
| | $a_3$ | 10 | 9 | 9 | 16 | 9* |
| | $a_4$ | 3 | 5 | 9 | 10 | 3 |
| | Column maxima | 16 | 9* | 9* | 16 | |

In this example, the optimal strategy for Alpha Limited is $a_3$ and the optimal strategies for Beta Limited are $b_2$ and $b_3$. There are two saddle points at 9 which is the value of the game. It means gain of 9% market share for Alpha Limited and loss of 9% market share for Beta Limited.

### 8.4.2 When Saddle Point does not exist

In case, saddle point does not exist, it is not possible to find the solution in terms of *pure strategies*- maximin and minimum strategy. The solution to such problems where saddle point does not exist calls for employing *mixed strategies*. A *mixed strategy* is the combination of two or more strategies selected by the players at a given time, according to a pre-determined probability. Players choose a mix of strategies in a given ratio.

Let's discuss the solution for 2 × 2 games where saddle point does not exist.

| B player's strategies | | | |
|---|---|---|---|
| A player's strategies | | $b_1$ | $b_2$ |
| | $a_1$ | 8 | -9 |
| | $a_2$ | -5 | 10 |

In the above problem, saddle point does not exist, so the method discussed in previous section will not suffice to find the optimal strategy for player A and B.

If A player choose $a_1$ strategy, then B player will choose $b_2$ and if A chooses $a_2$, then B player would choose $b_1$. So if B knows A's choice, then B can ensure his/her gain by choosing a strategy opposite to A. Therefore, A will make it difficult for B to guess what he/she is going to choose. Similarly, B will also make it difficult for A to guess the strategy B is likely to choose in the game situation. The players will play their strategy with ratio.

Now, A chooses strategy $a_1$ with probability x, then A will choose $a_2$ strategy with (1-x) probability. If player B plays $b_1$ strategy, then A's pay off can be determined with reference to the first column of the pay-off matrix as given below.

Expected pay-off of A if B adopts $b_1$ strategy = 8x- 5(1-x)

Similarly, expected pay-off of A if B adopts $b_2$ strategy = -9x + 10(1-x)

Now, if have to find the value of x so that the expected pay-off of A can be determined irrespective of the strategy adopted by B.

8x- 5(1-x) = -9x + 10(1-x)

8x-5+5x= -9x+10-10x

x= 15/32

This means A would adopt strategy $a_1$ and $a_2$ in the proportion of 15:17.

The expected pay-off for player A is

8x- 5(1-x) = ( 8 × 15/32 ) – ( 5 × 17/32 ) = 35/32

-9x + 10(1-x) = ( -9 × 15/32 ) + ( 10 × 17/32 ) = 35/32

Thus, player A will have a gain of 35/32 per play in the long run.

We can find out the mixed strategy for player B in similar manner. Now, B chooses strategy $b_1$ with probability y, then B will choose $b_2$ strategy with (1-y) probability. If player A plays $a_1$ strategy, then B's pay off can be determined with reference to the first row of the pay-off matrix as given below.

Expected pay-off of B if A adopts $a_1$ strategy = 8y- 9(1-y)

Similarly, expected pay-off of B if A adopts $a_2$ strategy = -5y + 10(1-y)

Now, if have to find the value of y so that the expected pay-off of B can be determined irrespective of the strategy adopted by A.

8y- 9(1-y) = -5y + 10(1-y)

8y- 9 + 9y = -5y + 10 -10y

y =19/32

This means B would adopt strategy $b_1$ and $b_2$ in the proportion of 19:13.

The expected pay-off (loss) for player B is

8y- 9(1-y) = ( 8 × 19/32 ) – ( 9 × 13/32 ) = 35/22

-5y + 10(1-y) = ( -5 × 19/32 ) + ( 10 × 13/32 ) = 35/22

This implies B will lose 35/22 per play in the long run.

The value of the game is 35/22.

| | Strategy | Ratio |
|---|---|---|
| Player A | $a_1$ | 15/32 |
| | $a_2$ | 17/32 |
| Player B | $b_1$ | 19/32 |
| | $b_2$ | 13/32 |

| B player's strategies | | | |
|---|---|---|---|
| A player's strategies | | $b_1$ | $b_2$ |
| | $a_1$ | $a_{11}$ | $a_{12}$ |
| | $a_2$ | $a_{21}$ | $a_{22}$ |

Formula:

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$V = \frac{a_{11} a_{22} - a_{12} a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

For the above example, Let's solve

$$x = \frac{10-(-5)}{(8+10)-(-9-5)} = 15/32$$

$$y = \frac{10-(-9)}{(8+10)-(-9-5)} = 19/32$$

$$V = \frac{(8 \times 10)-(-9 \times -5)}{(8+10)-(-9-5)} = 35/32$$

The values match with the solution obtained earlier. This means A would adopt strategy $a_1$ and $a_2$ in the proportion of 15:17. B would adopt strategy $b_1$ and $b_2$ in the proportion of 19:13. The value of the game is 35/32 which means player A gains and Player B loses 35/32.

### 8.4.3 Dominance Rule

In a game, a player may find one strategy to dominate over the other(s). This means that in all situations, a particular strategy will be preferred over the other(s). This concept of domination of strategy is extremely useful in simplifying the problem and finding the solution to the game.

Let's consider an example,

| | | B player's strategies | | |
|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ |
| A player's strategies | $a_1$ | 8 | -9 | 10 |
| | $a_2$ | -5 | 10 | 6 |
| | $a_3$ | 6 | -11 | 5 |

We notice that every element of first row exceeds the corresponding element of third row in the matrix (8 > 6; -9 > -11 and 10 > 5). This means that in any given situation, player A will always prefer $a_1$ over $a_3$. Thus, $a_1$ dominates $a_3$. Hence, $a_3$ can be deleted.

| B player's strategies | | | | |
|---|---|---|---|---|
| A player's strategies | | $b_1$ | $b_2$ | $b_3$ |
| | $a_1$ | 8 | -9 | 10 |
| | $a_2$ | -5 | 10 | 6 |

From the reduced matrix, we observe that every element of first column is greater than the corresponding element in third column. Since, B would like to minimize the pay-off for A, B will select $b_1$ over $b_3$ always. Hence, $b_1$ will dominate over $b_3$. Thus, $b_3$ can be eliminated.

| A player's strategies | | $b_1$ | $b_2$ |
|---|---|---|---|
| | $a_1$ | 8 | -9 |
| | $a_2$ | -5 | 10 |

Now, the problem is reduced to a 2× 2 and exactly same as the previous example. Thus, it can be solved in the manner explained earlier and the solution will be as follows.

The value of the game is 35/22.

| | Strategy | Ratio |
|---|---|---|
| Player A | $a_1$ | 15/32 |
| | $a_2$ | 17/32 |
| | $a_3$ | 0 |
| Player B | $b_1$ | 19/32 |
| | $b_2$ | 13/32 |

| | | |
|---|---|---|
| | $b_3$ | 0 |

### 8.4.4 Linear Combination

Let's consider the following game.

| B player's strategies | | | |
|---|---|---|---|
| A player's strategies | | $b_1$ | $b_2$ |
| | $a_1$ | 28 | 0 |
| | $a_2$ | 2 | 12 |
| | $a_3$ | 4 | 7 |

In this problem, we see no strategy is dominating over any other strategy. However, we notice that a linear combination of strategy $a_1$ and $a_2$ in the ration of 1:3 will always dominate strategy $a_3$ in all situations. Please note that the ratio in which combination of strategies will dominate another strategy is checked using trial and error method.

$\frac{1}{4} \times 28 + \frac{3}{4} \times 2 > 4$ and $\frac{1}{4} \times 0 + \frac{3}{4} \times 12 > 7$

The problem can be reduced 2 × 2 matrix as following:

Let's consider the following game.

| | | | |
|---|---|---|---|
| A player's strategies | | $b_1$ | $b_2$ |
| | $a_1$ | 28 | 0 |
| | $a_2$ | 2 | 12 |

$$x = \frac{a_{22} - a_{21}}{(a11 + a22) - (a12 + a21)} = \frac{12 - 2}{(28+12) - (0+2)} = 5/19$$

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{12-0}{(28+12) - (0+2)} = 6/19$$

$$V = \frac{a_{11}\,a_{22} - a_{12}\,a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{28 \times 12 - 0 \times 2}{(28+12) - (0+2)} = 336/38$$

The optimal strategy for A is (5/19 , 4/19, 0) and for B is (6/19, 13/19) and the game value is 168/19.

## 8.5 SOLUTION OF *M* × *N* GAMES – FORMULATION AND SOLUTION AS A LPP

Let's consider the following game.

| B player's strategies | | | | |
|---|---|---|---|---|
| A player's strategies | | $B_1$ | $B_2$ | $B_3$ |
| | $A_1$ | $a_{11}$ | $a_{12}$ | $a_{13}$ |
| | $A_2$ | $a_{21}$ | $a_{22}$ | $a_{23}$ |
| | $A_3$ | $a_{31}$ | $a_{32}$ | $a_{33}$ |

| B player's strategies | | | | |
|---|---|---|---|---|
| A player's strategies | | $B_1$ | $B_2$ | $B_3$ |
| | $A_1$ | 8 | 9 | 3 |
| | $A_2$ | 2 | 5 | 6 |
| | $A_3$ | 4 | 1 | 7 |

We can solve the above problem by formulating it as a LPP from A's and B's point of view. Let's first consider it from the point of A. We assume $x_1$, $x_2$, $x_3$ as the probabilities with which player A will choose strategies $A_1$, $A_2$ and $A_3$ respectively.

Player A would use *maximin* strategy which is to maximize the minimum gain from playing this game which is assumed as 'U'.

Now, the expected pay-off of A will be as follows:

$E_1 = a_{11}x_1 + a_{21}x_2 + a_{31}x_3$ (If player B chooses $B_1$)

$E_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3$ (If player B chooses $B_2$)

$E_3 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3$ (If player B chooses $B_3$)

Now, we can express the problem as

Maximize U (Maximize the minimum value of U)
subject to
$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \geq U$

$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \geq U$

$a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \geq U$

$x_1 + x_2 + x_3 = 1$

Now, assuming that U is positive (which would be if all pay-offs are positive), we can divide the constraints by U and attempt to minimize 1/U rather than maximize U.

We further define a new variable $X_i = x_i/U$ and restate the problem as follows.

Minimize $1/U = X_1 + X_2 + X_3$

$a_{11}X_1 + a_{21}X_2 + a_{31}X_3 \geq 1$

$a_{12}X_1 + a_{22}X_2 + a_{32}X_3 \geq 1$

$a_{13}X_1 + a_{23}X_2 + a_{33}X_3 \geq 1$

$X_1 , X_2 , X_3 \geq 0$

So, for the above problem, we can formulate the game situation as a LPP for player A's point of view.

Minimize $1/U = X_1 + X_2 + X_3$

$8X_1 + 2X_2 + 4X_3 \geq 1$

$9X_1 + 5X_2 + X_3 \geq 1$

$3X_1 + 6X_2 + 7X_3 \geq 1$

$X_1 , X_2 , X_3 \geq 0$

Now, we can simply solve the above LPP using Simplex method and obtain the solution.

If, we look at the problem from player B's point of view, We assume $y_1$, $y_2$, $y_3$ as the probabilities with which player B will choose strategies $B_1$, $B_2$ and $B_3$ respectively.

Player B would use *minixmax* strategy which is to minimize the maximum gain from playing this game which is assumed as 'V'.

Now, the expected pay-off of B will be as follows:

$E'_1 = a_{11}y_1 + a_{12}y_2 + a_{13}y_3$ (If player A chooses $A_1$)

$E'_2 = a_{21}y_1 + a_{22}y_2 + a_{23}y_3$ (If player A chooses $A_2$)

$E'_3 = a_{31}y_1 + a_{32}y_2 + a_{33}y_3$ (If player A chooses $A_3$)

Now, we can express the problem as

Minimixe V
subject to

$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \leq V$

$a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \leq V$

$a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \leq V$

$y_1 + y_2 + y_3 = 1$

We assume V is positive (which would be if all pay-offs are positive), we can divide the constraints by V and attempt to maximize 1/V rather than minimize V.

We further define a new variable $Y_i = y_i/V$ and restate the problem as follows.

Maximize $1/V = Y_1 + Y_2 + Y_3$

$a_{11}Y_1 + a_{12}Y_2 + a_{13}Y_3 \leq V$

$a_{21}Y_1 + a_{22}Y_2 + a_{23}Y_3 \leq V$

$a_{31}Y_1 + a_{32}Y_2 + a_{33}Y_3 \leq V$

$Y_1, Y_2, Y_3 \geq 0$

This is the dual of LPP given earlier.

So, for the above problem, we can formulate the game situation as a LPP for player B's point of view.

Maximize $1/V = Y_1 + Y_2 + Y_3$

$8Y_1 + 9Y_2 + 3Y_3 \leq 1$

$2Y_1 + 5Y_2 + 6Y_3 \leq 1$

$4Y_1 + Y_2 + 7Y_3 \leq 1$

$Y_1, Y_2, Y_3 \geq 0$

Now, we can simply solve the above LPP using Simplex method and obtain the solution.

We would be solving the maximization problem and reading the optimal solution of the primal (minimization problem) from the optimal solution of the dual.

Maximize $1/V = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + 0S_3$

$8Y_1 + 9Y_2 + 3Y_3 + S_1 = 1$

$2Y_1 + 5Y_2 + 6Y_3 + S_2 = 1$

$4Y_1 + Y_2 + 7Y_3 + S_3 = 1$

$Y_1, Y_2, Y_3, S_1, S_2, S_3 \geq 0$

Table 1

| $C_j$ | Basic variable | Basic Solution | $Y_1$ | $Y_2$ | $Y_3$ | $S_1$ | $S_2$ | $S_3$ | Ratio | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $S_1$ | 1 | 8 | 9 | 3 | 1 | 0 | 0 | 1/8 | ← |
| 0 | $S_2$ | 1 | 2 | 5 | 6 | 0 | 1 | 0 | 1/2 | |
| 0 | $S_3$ | 1 | 4 | 1 | 7 | 0 | 0 | 1 | 1/4 | |
| $C_j$ → | | | 1 | 1 | 1 | 0 | 0 | 0 | | |
| $Z_j$ | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| $C_j - Z_j$ | | | 1 | 1 | 1 | 0 | 0 | 0 | | |
| | | | ↑ | | | | | | | |

So, in table 2, $S_1$ exists and $Y_1$ enters.

Table 2

| $C_j$ | Basic variable | Basic Solution | $Y_1$ | $Y_2$ | $Y_3$ | $S_1$ | $S_2$ | $S_3$ | Ratio | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $Y_1$ | 1/8 | 1 | 9/8 | 3/8 | 1/8 | 0 | 0 | 1/3 | |
| 0 | $S_2$ | 3/4 | 0 | 11/4 | 21/4 | -1/4 | 1 | 0 | 1/7 | |
| 0 | $S_3$ | 1/2 | 0 | -7/2 | 11/2 | -1/2 | 0 | 1 | 1/11 | ← |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_j$ → | | | 1 | 1 | 1 | 0 | 0 | 0 | | |
| $Z_j$ | | 1/8 | 1 | 9/8 | 3/8 | 1/8 | 0 | 0 | | |
| $C_j - Z_j$ | | | 0 | -1/8 | 5/8 | -1/8 | 0 | 0 | | |
| | | | | | ↑ | | | | | |

So, in table 3, $S_3$ exists and $Y_3$ enters.

3Table 3

| $C_j$ | Basic variable | Basic Solution | $Y_1$ | $Y_2$ | $Y_3$ | $S_1$ | $S_2$ | $S_3$ | Ratio | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $Y_1$ | 1/11 | 1 | 15/11 | 0 | 7/44 | 0 | -3/44 | 1/15 | |
| 0 | $S_2$ | 3/11 | 0 | 67/11 | 0 | 5/22 | 1 | -21/22 | 3/67 | ← |
| 1 | $Y_3$ | 1/11 | 0 | -7/11 | 1 | -1/11 | 0 | 2/11 | -1/67 | |
| $C_j$ → | | | 1 | 1 | 1 | 0 | 0 | 0 | | |
| $Z_j$ | | | 1 | 8/11 | 1 | 3/44 | 0 | 5/44 | | |
| $C_j - Z_j$ | | | 0 | 3/11 | 0 | -3/44 | 0 | -5/44 | | |
| | | | | ↑ | | | | | | |

So, in table 3, $S_2$ exists and $Y_{32}$ enters.

Table 4

| $C_j$ | Basic variable | Basic Solution | $Y_1$ | $Y_2$ | $Y_3$ | $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $Y_1$ | 2/67 | 1 | 0 | 0 | 29/268 | -15/67 | 39/268 |
| 01 | $Y_2$ | 3/67 | 0 | 1 | 0 | 5/134 | 11/67 | -21/134 |
| 1 | $Y_3$ | 8/67 | 0 | 0 | 1 | -9/134 | 7/67 | 11/134 |

| $C_j \rightarrow$ | | | 1 | 1 | 1 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|
| $Z_j$ | | | 1 | 1 | 1 | 21/268 | 12/268 | 19/268 |
| $C_j - Z_j$ | | | 0 | 0 | 0 | -21/268 | -12/268 | -19/268 |

Optimal solution is obtained.

Subtituting values,

I/V = 2/67 +3/67 +8/67 = 13/67

Value of the Game V= 67/13

$Y_i = y_i/V$
so, $y_1 = 2/13$

$y_2 = 3/13$

$y_3 = 8/13$

We can read the optimal solution for the primal from the $C_j - Z_j$ values so $X_1$, $X_2$, $X_3$ are 21/268, 12/268 and 19/268 respectively.

U= 21/268 + 12/268 +19/268 = 13/67

$X_i = x_i/U$

So, $x_1 = 21/52$

$x_2 = 12/52$

$x_3 = 19/52$

The optimal strategy for player A is in the ratio of 21:12: 19 and for player B is 2:3:8. The value of the game = 67/13.

## IN-TEXT QUESTIONS

1. Saddle point exists when values from maximin and minimax strategy are ______________________.
2. Game situation occurs when two or more player have ______________________interests.
3. Every game situation must have a saddle point. True / False
4. The strategy which will always be preferred by a player over other strategies in any situation is called ________________ strategy.
5. A game situation where the gain of one player is equal to the loss of other player is called _____________.
6. In a two-person game, both players should have equal number of strategies. True/False
7. Saddle point is the point of equilibrium. True/False
8. The combination of strategies used by the player(s) in a particular ratio is called _____________ strategy.
9. Dominance principle implies that strategies of one player are dominating over the strategies of other player. True/False
10. Mixed strategy can use only combination of two strategies. True/False



## 8.6 SUMMARY

In this lesson, we learnt about decision making in game situations where players have conflicting interest and want to know the optimal strategy to be employed. The pay-off matrix is known to the players but their decisions are interdependent. We learnt about two-person zero sum games in different cases- when saddle point exists, when saddle point does not exist, dominance rule and linear combination. The Solution of $m \times n$ games is also discussed with the help of formulation and solution as a LPP

## 8.7 GLOSSARY

- **Game:** The situation of conflicting interests among the opponents is called a *game.*
- **Strategy:** A *strategy* is the action taken by the player in various game situations.
- **Pay-off:** Each strategy chosen by the player in a game situation leads to outcomes called *pay-offs.*
- **Saddle point:** The point of equilibrium where the maximin value is equal to the minimax value is called *saddle point.*
- **Mixed Strategy:** A *mixed strategy* is the combination of two or more strategies selected by the players at a given time, according to a pre-determined probability.
- **Dominance Rule:** In a game, a player may find one strategy to dominate over the other(s). This means that in all situations, a particular strategy will be preferred over the other(s) by the player.



## 8.8 ANSWERS TO IN-TEXT QUESTIONS

| | |
|---|---|
| 1. same/equal | 6. False |
| 2. conflicting/contradicting | 7. True |
| 3. False | 8. Mixed |
| 4. dominating | 9. False |
| 5. zero sum game | 10. False |

## 8.9 SELF-ASSESSMENT QUESTIONS

1. Solve the following game and determine the value of the game and optimal strategies for both the players.

| B player's strategies | | | | | |
|---|---|---|---|---|---|
| A player's strategies | | $B_1$ | $B_2$ | $B_3$ | $B_3$ |
| | $A_1$ | 3 | 2 | 4 | 0 |
| | $A_2$ | 3 | 4 | 2 | 4 |
| | $A_3$ | 4 | 2 | 4 | 0 |
| | $A_4$ | 0 | 4 | 0 | 8 |

2. Solve the following game and determine the value of the game and optimal strategies for both the players.

| B player's strategies | | | | |
|---|---|---|---|---|
| A player's strategies | | $B_1$ | $B_2$ | $B_3$ |
| | $A_1$ | 3 | -1 | 4 |
| | $A_2$ | 6 | 7 | -2 |

3. Solve the following game and determine the value of the game and optimal strategies for both the players.

| B player's strategies | | | |
|---|---|---|---|
| A player's strategies | | $B_1$ | $B_2$ |
| | $A_1$ | 3 | 7 |
| | $A_2$ | -5 | 5 |

4. Solve the following game and determine the value of the game and optimal strategies for both the players.

| B player's strategies | | | | |
|---|---|---|---|---|
| A player's | | $B_1$ | $B_2$ | $B_3$ |
| | $A_1$ | 5 | 9 | 3 |

| | | | | |
|---|---|---|---|---|
| strategies | $A_2$ | 6 | -12 | -11 |
| | $A_3$ | 8 | 16 | 10 |

5. Solve the following game and determine the value of the game and optimal strategies for both the players.

| B player's strategies | | | | |
|---|---|---|---|---|
| A player's strategies | | No promotion | Medium promotion | High promotion |
| | No promotion | 5 | 9 | 3 |
| | Medium promotion | 6 | -12 | -11 |
| | High promotion | 8 | 16 | 10 |

## 8.10 REFERENCES & SUGGESTED READINGS


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